Imperial College of Science, Technology

and Medicine

Department of Mathematics

Convexity Adjustments in Inflation–linked Derivatives using a multi-factor version of the Jarrow and Yildirim (2003) Model

Hongyun Li

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Abstract

In this paper, we use a Gaussian HJM-type (Heath et al 1992) model for the valuation of inflation-linked derivatives. The model is essentially that of Jarrow and Yildirim (2003), which in turn is essentially analogous to a cross-currency model (modelling the spot foreign exchange rate, domestic currency interest-rates and foreign currency interest-rates). In the cross-currency FX analogy of Jarrow and Yildirim (2003), nominal zero coupon bonds are analogous to zero coupon bonds in the domestic currency, real zero coupon bonds are analogous to zero coupon bonds in the foreign currency and the spot consumer price index (CPI) is analogous to the spot foreign exchange rate.

We extend the Jarrow and Yildirim (2003) model by modelling interest-rate yield curves with a multi-factor (rather than one factor) Gaussian HJM (Heath et al 1992) model. Our paper is organized as follows:

Firstly, we introduce the model and our notation.

Then, we explain the valuation of standard zero coupon inflation swaps. We then examine popular and actively-traded inflation products such as zero coupon inflation swaps with delayed payment, period-on-period inflation swaps with both no delayed payments and with delayed payments, using the Gaussian model (we explain what we mean by delayed payments in section 2.4). Moreover, we give the analytical prices of these inflation-linked derivatives, consistent with no-arbitrage. Specifically, we focus on the "convexity adjustments" involved in pricing these products. We provide an application of our convexity adjustment formulae to the valuation of limited price indexation (LPI) swaps.

Finally, to specify the model, we use the same method as in Jarrow and Yildirim (2003) to estimate the model parameters which are needed to evaluate the convexity adjustments.

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TABLE OF CONTENTS

1 Introduction

1.1 Background to inflation-linked derivatives

In recent years, the market for inflation-linked derivatives has grown very rapidly.

They are used by market participants to manage the risks of changing inflation and changing inflation expectations in an efficient way. It is fair to say that inflation is now regarded as an independent asset class.

There are, broadly speaking, two types of participants in the inflation derivatives markets: those who wish to receive and those who wish to pay inflation-linked cash flows.

Actively-traded inflation derivatives include standard zero coupon inflation swaps, and more complicated products such as period-on-period inflation swaps (Mercurio (2005)), inflation caps, inflation swaptions, and futures contracts written on inflation (Crosby 2007b).

Inflation is described in terms of an inflation index. In practice, there are a number of actively referenced inflation indices. The main indices are the HICPxT index (which measures inflation in the Euro zone and is published by Euro stat), the RPI (Retail Price index) (which measures inflation in the UK and is published by National Statistics), and the US-CPI (consumer price index) (which measures inflation in the US and is published by BLS).

Throughout this paper, we will, for the sake of brevity, refer to the inflation index as the CPI index or the spot CPI index (even though in the UK, it would be probably the RPI and in the Eurozone, it would probably be the HICPxT index). All of these indices are a measure of retail or consumer price inflation. They are calculated by collecting and comparing the prices of a set basket of goods and services, as bought by a typical consumer, at regular intervals over time. (Reuters)

1.2 Outline of the thesis

The remainder of this paper is structured as follows:

In chapter two, the foreign exchange analogy is explained briefly. Then, we will provide notation and discuss the simplest type of inflation derivative, namely standard (i.e. with no delayed payment) zero coupon inflation swaps, and show how they can be valued in a model-independent fashion. Moreover, we will explain in detail what is meant by indexation lag and delayed payments.

In chapter three, working within a multi-factor version of the Jarrow and Yildirim (2003) model, we will introduce the dynamics of nominal zero coupon bond prices, real zero coupon bond prices and the spot CPI index.

In chapter four, we compute the convexity adjustments required to value period-on-period inflation swaps with no delayed payments, zero coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments in detail. To our best knowledge, some of these results, at least in the context of a multi-factor Jarrow and Yildirim (2003) model, have not appeared in the literature before.

In chapter five, we provide an application of the convexity adjustments we computed in chapter four, to the valuation of limited price indexation (LPI) swaps, in which we use the quasi-analytic methodology of Ryten (2007).

In chapter six, we use the methods of Jarrow and Yildirim (2003) to estimate the model parameters from historical data. We will also illustrate our model with some examples and comparisons.

In chapter seven, we will give the conclusions of this paper.

In the appendices, we give detailed derivations of some of the formulae that we use.

2 Foreign Exchange Analogy and Modeling Inflation

We are concerned, in this paper, with Gaussian models for inflation which are arbitrage-free and consistent with any initial term structure of interest-rates (both nominal and real).

In this and all subsequent sections, we will always make the assumptions that the market is frictionless, complete and arbitrage-free. These assumptions guarantee (Harrison and Pliska (1981)) the existence of a unique equivalent martingale measure which is denoted by Q. We use the notation $E_t[\ ]$ to denote expectations at time t, with respect to this equivalent martingale measure.

All stochastic processes are defined on a common filtered probability space (Ω, F, Q), where the filtration F is assumed to be the natural filtration generated by the Brownian motions, which we shall shortly introduce, driving the nominal and real interest-rate yield curves and the spot CPI index.

We denote calendar time by t. We define today (the initial time) to be time t_0.

2.1 Introduction to foreign exchange methodology

2.11 Notation

The foreign exchange (FX) analogy relates to the valuation of foreign exchange options written on a spot foreign exchange rate.

We denote the price, in domestic currency, of a (credit risk free) zero coupon bond denominated in domestic currency, at time t, maturing at time T by $P(t,T)$, and the corresponding domestic short rate by $r(t)$. We denote the price, in foreign currency, of a (credit risk free) zero coupon bond denominated in foreign currency, at time t, maturing at time T by $P_f(t,T)$, and the corresponding foreign short rate by $r_f(t)$. Let $X(t)$ denote the spot foreign exchange (FX) rate, at time t, quoted as the number of units of domestic currency per unit of foreign currency.

2.12 A basic valuation formula in the foreign exchange analogy

The basic derivatives valuation formula is (Harrison and Pliska (1981)): The price, H_{t_0}, of a derivative, at time t_0, is: $H_{t_0} = E_{t_0}\left[\exp\left(-\int_{t_0}^{T} r(s)ds\right)H_T\right]$, where H_T is the random payout at time T.

There is a key observation for modelling cross-currency derivatives which is:

$$E_t\left[\exp\left(-\int_t^{T_M} r(s)ds\right)X(T_M)\right] = X(t)P_f(t,T_M) \tag{2.11}$$

which we will refer to later in section 2.22.

Remark: The above equation is true and model-independent. To see that it is true, observe that, $P_f(t,T_M)$ is the price of a zero coupon bond in foreign currency, at time t maturing at time T_M, and $X(t)$ is the price, in domestic currency, of one unit of foreign currency paid at time t.Therefore the RHS of the equation represents the price, at time t, in domestic currency, of one unit of foreign currency paid at time T_M. In terms of the LHS, since $X(T_M)$ denotes the price, in domestic currency, of one unit of foreign currency paid at time T_M, then the conditional expectation of it, discounted to time t, represents the price, at time t, in domestic currency, of one unit of foreign money paid at time T_M. Hence, the equation must be true.

2.2 Modeling Inflation

The key to modeling inflation and to pricing inflation-linked derivatives is to notice that there is a total and complete analogy between inflation-linked derivatives and cross-currency derivatives.

The analogy is that nominal interest rates are the equivalent of domestic interest rates, real interest rates are the equivalent of foreign interest rates and the spot CPI inflation index is the equivalent of the spot foreign exchange rate. The FX analogy gives an intuitive way to think about inflation. See the figure below

Table 1 The analogy

FX rate $X(t)$	CPI index $X(t)$
Domestic interest rate $r(t)$	Nominal interest rate $r(t)$
Foreign interest rate $r_f(t)$	Real interest rate $r_r(t)$

2.21 Notation

Let us explain some notation. We will use a subscript r to indicate real interest rates and real zero coupon bond prices.

Let $r(t)$ and $r_r(t)$ denote the (continuously compounded) risk-free nominal and real short rates, at time t, respectively. Let $P(t,T)$ and $P_r(t,T)$ denote the price of a (credit risk free) nominal and real zero coupon bond, at time t, maturing at time T, respectively. Throughout this paper, we will often use the words "zero coupon bond" and "discount factor" almost interchangeably, with the proviso that discount factors are known today, time t_0.

Let $X(t)$ denote the spot CPI index, at time t, i.e. it is the value, in units of nominal currency, of a typical basket of goods and services.

2.22 An Important Observation for inflation derivatives

The key observation for pricing inflation derivatives is that, for any times t and T_M, with $t \le T_M$, we have:

$$E_t\left[\exp\left(-\int_t^{T_M} r(s)\,ds\right)X(T_M)\right] = X(t)P_r(t,T_M) \tag{2.21}$$

Remark: The above equation is model independent. It is the analogous equation to (2.11) for modeling inflation derivatives. To see that it is true, note that $P_r(t,T_M)$ is the price of a real zero coupon bond, at time t, maturing at time T_M, and $X(t)$ is the price, in nominal currency, at time t, of one unit of real currency paid at time t. Therefore the RHS of the equation represents the price, at time t, in nominal currency, of one unit of real currency paid at time T_M. In terms of the LHS, since $X(T_M)$ denotes the price, in nominal currency, of one unit of real currency paid at time T_M, then the conditional expectation of it discounted to time t, represents the price, at time t, in nominal currency, of one unit of real currency paid at time T_M. Hence, intuitively, equation (2.21) holds.

Remark: In Appendix 2, we give a more mathematical proof of equation (2.21). We can use this key observation to help price several different types of inflation-linked derivatives, including zero coupon inflation swaps and period-on-period inflation swaps, which we will explain in detail later.

2.3 Zero Coupon Inflation Swaps

Suppose that today, time t_0, we enter into a T_M year standard zero coupon inflation swap. As with a standard interest-rate swap, there is no up-front cost to entering into a zero coupon inflation swap. So the value of the swap today, time t_0, must be zero. The exchange of cash flows between the two parties only occurs at the maturity T_M of the swap.

We wish to value the swap, at time t, where $t_0 \le t \le T_M$. By definition, the payoff of the zero coupon inflation swap at time T_M is:

$$N\left(\frac{X(T_M)}{X(t_0)}-1\right)-N\left((1+K)^{T_M}-1\right) \tag{2.31}$$

where K is the fixed rate on the swap and N is the notional amount. We can call the first term $N\left(X(T_M)/X(t_0)-1\right)$ in the expression (2.31) the floating (inflation-linked) side and the second term $N\left((1+K)^{T_M}-1\right)$ the fixed side.

In the absence of arbitrage, the value of the swap, at time t, is:

$$\begin{aligned}
&E_t\left[\exp\left(-\int_t^{T_M} r(s)\,ds\right)\left(N\left(\frac{X(T_M)}{X(t_0)}-1\right)-N\left((1+K)^{T_M}-1\right)\right)\right]\\
&=E_t\left[N\exp\left(-\int_t^{T_M} r(s)\,ds\right)\left(\frac{X(T_M)}{X(t_0)}-(1+K)^{T_M}\right)\right]\\
&=\frac{N}{X(t_0)}E_t\left[\exp\left(-\int_t^{T_M} r(s)\,ds\right)X(T_M)\right]-NP(t,T_M)(1+K)^{T_M}\\
&=\frac{N}{X(t_0)}X(t)P_r(t,T_M)-NP(t,T_M)(1+K)^{T_M}
\end{aligned}$$

where in the last line, we have used the key observation, equation (2.21).

So the value of the swap, at time t, is:

$$\frac{N}{X(t_0)} X(t) P_r(t,T_M) - NP(t,T_M)(1+K)^{T_M} \tag{2.32}$$

which gives us a valuation formula for the value of the swap at any time.

In particular, we know that the value of the swap today, time t_0, must be zero. So setting $t = t_0$ in equation (2.32) and equating the value to zero implies:

$$0 = \frac{N}{X(t_0)} X(t_0) P_r(t_0,T_M) - NP(t_0,T_M)(1+K)^{T_M}.$$

Therefore,
$$P_r(t_0,T_M) = P(t_0,T_M)(1+K)^{T_M} \tag{2.33}$$

Remark: Zero coupon inflation swaps are actively traded in the market and one can get prices in the brokers. They are quoted by the fixed rate K for various maturities T_M. Hence, we can use the last equation to obtain the real interest-rate yield-curve i.e. obtain a set of real interest-rate discount factors (given a set of nominal interest-rate discount factors which, of course, we can get in the usual way from the standard interest-rate swaps market), which we can then use to price more exotic structures such as period-on-period swaps.

Remark: Comparing the methodology of Jarrow and Yildirim (2003), in which they use a stripping method to get nominal and real zero coupon bond prices from the observed market prices of coupon bearing bonds, it is much easier and quicker to get real discount factors from equation (2.33). Note that, in practice, T_M is usually a whole number of years. This means we obtain a set of real interest-rate discount factors to times which are a whole number of years from today. When interpolating between these times, to estimate real discount factors to times which are fractional numbers of years, one needs to be aware of the impact of seasonality. We will not discuss seasonality further here but we refer the reader to Belgrade and Benhamou (2004) and Kerkhof (2005).

Remark: Equations (2.32) and (2.33) are model independent and are not based on specific assumptions concerning the evolution of interest rate yield curves or the spot CPI index, but, indeed, simply follow from the absence of arbitrage.

2.4 Indexation lag and delayed payments

The main purpose of inflation-linked derivatives is to protect the real (i.e. after allowing for inflation) value of future cash flows. In order to achieve a high degree of certainty in the real value of future cash flows, the inflation-linked cash flows should be as closely linked as possible to contemporaneous inflation. However, this is not completely possible owing to the existence of indexation lag. This is best explained as follows:

In practice, there is a delay of a few weeks between the date on which the CPI index is measured and the date on which the value of the CPI index is announced in the market. This time interval is the time required to collect and process the consumer prices required by statisticians to compute the CPI index. For example, in the United Kingdom, the value of the CPI index (actually, one of the most closely watched indices is called the RPI but we shall continue to call it the CPI for brevity) for a given month is published on about the 15th of the following month. So for example, the CPI index for May 2007 was published on about the 15th June 2007. Furthermore, the market for sterling denominated zero coupon inflation swaps adopts the convention that throughout a calendar month, the “base” index (the value of the index appearing in the denominator of the payoff) is the index for two months before. So, for example, throughout July 2007, all 25 year zero coupon inflation swaps would have a future inflation-linked payoff (in July 2032) equal to:

The value of the CPI index for May 2032 (which will be announced in June 2032) divided by the value of the CPI index for May 2007 (whose value was known on approximately 15th June 2007) minus one.

This means that an investor who receives the inflation-linked payment on a 25 year zero coupon swap is not compensated for inflation over the period May to July 2032 although the investor will receive compensation for inflation over the period May to July 2007 (before the swap commenced).

When we write $X(t)$ as the value of the spot CPI index, what we really mean is that $X(t-\varepsilon)$ is the actual published value of the CPI index at a time ε earlier. The value of ε can actually vary slightly (between about one month and two months). Since the value of ε only changes slightly compared to the typically maturity of inflation swaps (which is often greater than 20 years), it is the market convention to assume that it is effectively constant. This is the convention we will adopt. There is very little to be lost by doing so since the same convention applies at the maturity of the inflation swap as applied at the start of the inflation swap and so any misspecification, at least partially, cancels out. One convenient benefit of adopting this convention is that we can continue to use equation (2.33) to obtain real interest-rate discount factors and to do so in a model-independent fashion.

There is one further issue with inflation swaps which is the issue of delayed payments. This is sometimes called payment lag although to avoid confusion with the concept of indexation lag, we will refer to it as delayed payments.

For standard zero coupon inflation swaps, the payment time T_M of the payoff coincides with the argument of the value of X in the numerator of the inflation-linked term in the payoff. So, the payment of the cash flow in equation (2.31), namely, $N\left(X\left(T_M\right)/X\left(t_0\right)-1\right)-N\left(\left(1+K\right)^{T_M}-1\right)$, occurs at time T_M. Although, this is indeed the most common situation, often, in practice, the payment is delayed until some later time T_N. This delay is not just the standard 2 day spot settlement lag but can be a period of a few weeks, a few months or even several years. We will refer to such inflation swaps as inflation swaps with delayed payments.

To see how such inflation swaps have an important economic rationale, consider a commercial property company. Suppose it has debt in the form of fixed-rate loans. It receives rents from its tenants which it wants to pay out as the inflation-linked leg of an inflation swap. It will receive fixed payments on the inflation swap which it will use to pay its fixed-rate debt. Often rents will remain constant for a period of 5 years before being reviewed. They will then be revised upwards to reflect inflation over those intervening five years. So for example, suppose, the commercial property company wanted to enter into an inflation swap trade, in which it paid inflation-linked cash flows and it received fixed cash flows. The company wants to hedge the cash flows that it will receive from its tenants in years 6, 7, 8, 9 and 10. So a suitable inflation swap trade would be a strip of five zero coupon inflation swaps as follows. The payoff of the five zero coupon swaps would be (we write only the inflation-linked leg with unit notional):

At the end of year 6, the company pays $\left(X(5)/X(0)-1\right)$. At the end of year 7, it again pays $\left(X(5)/X(0)-1\right)$. At the end of year 8, it again pays $\left(X(5)/X(0)-1\right)$. At the end of year 9, it again pays $\left(X(5)/X(0)-1\right)$. At the end of year 10, it again pays $\left(X(5)/X(0)-1\right)$.

We can see that these are zero coupon inflation swaps with delayed payment with the delay on the final swap of the strip being 5 years.

Period-on-period swaps with delayed payments also trade in the markets.

If nominal interest-rates were deterministic, then valuing these inflation swaps with delayed payments would be trivial given a pricing methodology for valuing the corresponding type of inflation swap with no delayed payments. However, since we will have stochastic interest-rates, valuation is more difficult and will involve the evaluation of additional terms which we will loosely refer to as convexity adjustments.

3 Framework of the model

In this section, we set up the dynamics of nominal zero coupon bond prices, real zero coupon bond prices and the spot CPI index. We work within a multi-factor version of the Jarrow and Yildirim (2003) model. It is clear that the Jarrow and Yildirim (2003) model is a model which is, firstly, arbitrage-free and, secondly, consistent with any initial term structure of nominal and real interest rates, since it is a HJM (Heath et al (1992)) model.

3.1 Stochastic evolution of nominal bond prices

We assume that, under the equivalent martingale measure defined with respect to the nominal money market account numeraire, nominal zero coupon bond prices are stochastic and follow a Gaussian HJM model (Heath et al. 1992):

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + \sum_{k=1}^{K_n} \sigma_{kn}(t,T)\,dz_{kn}(t). \tag{3.11}$$

where K_n is the number of Brownian motions, $dz_{kn}(t)$, for each k, $k=1,...,K_n$, denotes standard Brownian increments. Furthermore, the correlation between $dz_{jn}(t)$ and $dz_{kn}(t)$ is ρ_{jnkn}, for each k and each j, $j=1,...,K_n$, and $\sigma_{kn}(t,T)$, for each k, are volatility terms which are purely deterministic functions of t and T, satisfying $\sigma_{kn}(T,T) \equiv 0$.

3.2 Stochastic evolution of real bond prices

We now describe the risk-neutral dynamics of real zero coupon bond prices. We assume that, under the equivalent martingale measure defined with respect to the nominal money market account numeraire, real zero coupon bond prices follow a Gaussian HJM model (Heath et al.(1992)):

$$\frac{dP_r(t,T)}{P_r(t,T)} = \left(r_r(t) - \sum_{k=1}^{K_r} \rho_{krX}\sigma_X(t)\sigma_{kr}(t,T) \right) dt + \sum_{k=1}^{K_r} \sigma_{kr}(t,T)\,dz_{kr}(t) \tag{3.21}$$

where K_r is the number of Brownian motions, $dz_{kr}(t)$, for each k, $k=1,...,K_r$, denotes standard Brownian increments and where, for each k, $k=1,...,K_r$, ρ_{krX} is the correlation between the spot CPI and the respective Brownian motion driving real zero coupon bond prices. We denote the correlation between $dz_{jr}(t)$ and $dz_{kr}(t)$ by ρ_{jrkr}, for each k and each j, $j=1,...,K_r$.

Remark: Note the "quanto drift adjustment" in equation (3.21).

3.3 Stochastic evolution of the spot CPI index

The dynamics of the spot CPI, under the equivalent martingale measure defined with respect to the nominal money market account numeraire, are given by:

$$\frac{dX_t}{X_t} = \left(r(t) - r_r(t)\right)dt + \sigma_X(t)\,dz_X(t) \tag{3.31}$$

where $dz_X(t)$ denotes standard Brownian increments, the drift is the difference between the nominal and real short rates, and $\sigma_X(t)$ is the volatility which we assume to be a purely deterministic function of t. Furthermore, we introduce the notation that the correlation between dz_X and dz_{kn}, for each k, $k=1,...,K_n$, is ρ_{knX} and the correlation between dz_{kn} and dz_{jr}, for each k, $k=1,...,K_n$, and for each j, $j=1,...,K_r$ is ρ_{knjr}.

Remark: It is convenient to assume all the correlations are constant (which we do in the implementation) although all the equations in this paper would hold if they are, at most, deterministic functions of t. We assume the correlations form a symmetric positive-definite matrix with elements unity down the leading diagonal.

If we define the forward CPI at time t to (i.e. for the delivery at) time T by $F_X(t,T)$, then by no-arbitrage arguments (see Appendix 2), we know that,

$$F_X(t,T) = \frac{X_t P_r(t,T)}{P(t,T)} \quad . \tag{3.32}$$

Further, by Ito's lemma,

$$\begin{aligned}\frac{dF_X(t,T)}{F_X(t,T)} = &\left\{\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{knjn}\sigma_{jn}(t,T)\sigma_{kn}(t,T) - \sum_{k=1}^{K_n}\rho_{knX}\sigma_X(t)\sigma_{kn}(t,T)\right.\\ &\left. -\sum_{k=1}^{K_r}\sum_{j=1}^{K_n}\rho_{jnkr}\sigma_{jn}(t,T)\sigma_{kr}(t,T)\right\}dt\\ &+\sigma_X(t)dz_X(t) + \sum_{k=1}^{K_r}\sigma_{kr}(t,T)dz_{kr}(t) - \sum_{k=1}^{K_n}\sigma_{kn}(t,T)dz_{kn}(t)\end{aligned} \tag{3.33}$$

Then the forward CPI index, $F_X(t,T)$, at time t, can be expressed in terms of its value $F_X(t_0,T)$, at time t_0, as follows:

$$\begin{aligned}F_X(t,T) = &F_X(t_0,T)\\ &\times\exp\left(\int_{t_0}^{t}\left(\sigma_X(s)dz_X(s) + \sum_{k=1}^{K_r}\sigma_{kr}(s,T)dz_{kr}(s) - \sum_{k=1}^{K_n}\sigma_{kn}(s,T)dz_{kn}(s)\right)\right)\\ &\times\exp\left(\int_{t_0}^{t}\left(\frac{1}{2}\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{jnkn}\sigma_{jn}(s,T)\sigma_{kn}(s,T) - \frac{1}{2}\sum_{k=1}^{K_r}\sum_{j=1}^{K_r}\rho_{krjr}\sigma_{jr}(s,T)\sigma_{kr}(s,T)\right.\right.\\ &\left.\left.-\frac{1}{2}\sigma_X^2(s) - \sum_{k=1}^{K_r}\rho_{krX}\sigma_X(s)\sigma_{kr}(s,T)\right)ds\right)\end{aligned} \tag{3.34}$$

Remark: Notice that the drift and volatility terms in the stochastic differential equation for $F_X(t,T)$ are deterministic and that $F_X(t,T)$ is log-normally distributed.

4 Exotic inflation derivatives

In chapter two, we have shown that, given the rates on standard (i.e. with no delayed payment) zero coupon inflation swaps quoted in the market (and given nominal discount factors), we can get real discount factors.

We were able to obtain real discount factors by valuing zero coupon inflation swaps in a model-independent fashion. This is analogous to obtaining nominal discount factors from LIBOR deposit rates and by “bootstrapping” swap rates, which can also be done in a model-independent fashion. Just as nominal discount factors are the building blocks upon which we could value more exotic interest-rate derivatives, so real discount factors are the building blocks upon which we can value more exotic inflation derivatives. This is the aim of this section.

We will see that the prices of these more exotic inflation derivatives are model-dependent and therefore we will aim to value them in the Jarrow and Yildirim (2003) model we introduced in the last section. In this section, we will value three types of inflation swap, namely, period-on-period inflation swaps with no delayed payments, zero coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments.

The key point about the last two types of inflation swap is that they have the same payoff as the corresponding inflation swap with no delayed payments but the payoff is paid at a later time. When the delay in payment is very small (for example, a few weeks), we would, intuitively, expect the difference between the values of the corresponding swaps with no delayed payments and with delayed payments to be small. Conversely, we shall see that the difference in values can be substantial when the delay in payments is, for example, a few years. As we noted in section 2.4, inflation swaps with delayed payments of five years or more are quite commonly traded in the markets.

We now turn our attention to pricing period-on-period inflation swaps with no delayed payments.

4.1 Period-On-Period Inflation Swaps

Suppose that today, time t_0, we enter into a period-on-period inflation swap. The swap is defined via a set of fixed dates $T_0 < T_1 < T_2 < ... < T_{M-1} < T_M$, where $T_0 \equiv t_0$. These dates are usually approximately equally spaced (for example, approximately one year apart) but they need not be.

As with a standard interest-rate swap, a period-on-period inflation swap is made up of a series of swaplets. There are payments of $N\tau_{i,\inf}\left(X(T_i)/X(T_{i-1})-1\right)$ against a fixed rate $N\tau_{i,\,fixed}K$ at each time T_i.

Therefore, the payoff of the i^{th} swaplet, for $i = 1,2,...,M$, at time T_i is:

$$N\tau_{i,\inf}\left(\frac{X(T_i)}{X(T_{i-1})}-1\right)-N\tau_{i,\,fixed}K \qquad (4.11)$$

where K is the fixed rate on the swap, N is the notional amount, $\tau_{i,\inf}$ is the day-count adjusted time from T_{i-1} to T_i for the floating (inflation-linked) leg and $\tau_{i,\,fixed}$ is the day-count adjusted time from T_{i-1} to T_i for the fixed leg.

In the absence of arbitrage, the value of the swaplet, at time t, is :

$$\begin{aligned}
&E_t\left[\exp\left(-\int_t^{T_i} r(s)\,ds\right)\left(N\tau_{i,\inf}\left(\frac{X(T_i)}{X(T_{i-1})}-1\right)-N\tau_{i,\,fixed}K\right)\right] \\
&= N\tau_{i,\inf}E_t\left[\exp\left(-\int_t^{T_i} r(s)\,ds\right)\frac{X(T_i)}{X(T_{i-1})}\right]-P(t,T_i)\,N\left(\tau_{i,\inf}+\tau_{i,\,fixed}K\right) \qquad (4.12)
\end{aligned}$$

To value the floating (inflation-linked) side, we have to consider separately two different cases depending upon whether $t \geq T_{i-1}$ or $t < T_{i-1}$.

First case: $T_{i-1} \leq t \leq T_i$.

In this case, $X(T_{i-1})$ is known at time t. Therefore we can take $X(T_{i-1})$

outside of the expectation and then use the key observation, namely equation (2.21), and write $E_t\left[\exp\left(-\int_t^{T_i} r(s)\,ds\right)\frac{X(T_i)}{X(T_{i-1})}\right]=\frac{P_r(t,T_i)X(t)}{X(T_{i-1})}$. Hence, equation (4.12) becomes:

$$N\tau_{i,\inf}\frac{P_r(t,T_i)X(t)}{X(T_{i-1})}-P(t,T_i)N\left(\tau_{i,\inf}+\tau_{i,fixed}K\right) \tag{4.13}$$

Second case: $t_0 \le t < T_{i-1}$.

Using the law of iterated expectations in equation (4.12), we can write the value of the swaplet, at time t, as

$$N\tau_{i,\inf}E_t\left[\exp\left(-\int_t^{T_{i-1}} r(s)\,ds\right)E_{T_{i-1}}\left[\exp\left(-\int_{T_{i-1}}^{T_i} r(s)\,ds\right)\frac{X(T_i)}{X(T_{i-1})}\right]\right]-P(t,T_i)N\left(\tau_{i,\inf}+\tau_{i,fixed}K\right)$$

But now the key observation of equation (2.21), tells us that

$$\begin{aligned}E_{T_{i-1}}\left[\exp\left(-\int_{T_{i-1}}^{T_i} r(s)\,ds\right)\frac{X(T_i)}{X(T_{i-1})}\right]&=\frac{1}{X(T_{i-1})}E_{T_{i-1}}\left[\exp\left(-\int_{T_{i-1}}^{T_i} r(s)\,ds\right)X(T_i)\right]\\&=\frac{1}{X(T_{i-1})}P_r(T_{i-1},T_i)X(T_{i-1})\\&=P_r(T_{i-1},T_i)\end{aligned}$$

Therefore, the value of the swaplet, at time t, is :

$$N\tau_{i,\inf}E_t\left[\exp\left(-\int_t^{T_{i-1}} r(s)\,ds\right)P_r(T_{i-1},T_i)\right]-P(t,T_i)N\left(\tau_{i,\inf}+\tau_{i,fixed}K\right) \tag{4.14}$$

We can show (using the methodology of Appendix 4) that:

$$E_t\left[\exp\left(-\int_t^{T_{i-1}} r(s)\,ds\right)P_r(T_{i-1},T_i)\right]=P(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\exp\left(\int_t^{T_{i-1}} A(s,T_{i-1},T_i)\,ds\right) \tag{4.15}$$

where

$$\begin{aligned}\int_t^{T_{i-1}} A(s,T_{i-1},T_i)ds&=\sum_{k=1}^{K_n}\sum_{j=1}^{K_r}\int_t^{T_{i-1}}\rho_{knjr}\sigma_{kn}(s,T_{i-1})\left\{\sigma_{jr}(s,T_i)-\sigma_{jr}(s,T_{i-1})\right\}ds\\&+\sum_{k=1}^{K_r}\sum_{j=1}^{K_r}\int_t^{T_{i-1}}\rho_{krjr}\sigma_{kr}(s,T_{i-1})\left\{\sigma_{jr}(s,T_{i-1})-\sigma_{jr}(s,T_i)\right\}ds\\&+\sum_{k=1}^{K_r}\int_t^{T_{i-1}}\rho_{krX}\sigma_X(s)\left\{\sigma_{kr}(s,T_{i-1})-\sigma_{kr}(s,T_i)\right\}ds\end{aligned} \tag{4.16}$$

Remark: We call the expression (4.16) the convexity adjustment for a period-on-period inflation swap. Notice when $\sigma_{kr}\left(s,T_i\right)\equiv 0$, for all k, the convexity adjustment will be identically equal to zero. But, in this special case, real interest rates would be deterministic.

From equation (4.14) and (4.15), when $t_0 \le t < T_{i-1}$, the equation (4.12) becomes:

$$N\tau_{i,\inf} P\left(t,T_{i-1}\right)\frac{P_r\left(t,T_i\right)}{P_r\left(t,T_{i-1}\right)}\exp\left(\int_t^{T_{i-1}} A\left(s,T_{i-1},T_i\right)ds\right)-P\left(t,T_i\right)N\left(\tau_{i,\inf}+\tau_{i,fixed}K\right) \quad (4.17)$$

We can value a period-on-period inflation swap by summing up the value of all the swaplets, being careful to use equation (4.13) when $T_{i-1}\le t\le T_i$, and equation (4.17) when $t_0 \le t < T_{i-1}$.

We know that the value of the swap today, time t_0, must be zero. So we can set $t=t_0$ in the last formula and equate the value of the swap to zero, to relate the fixed rate K to the term structure of real interest-rate discount factors and to the parameters of the stochastic processes for the interest-rate yield curves and the spot CPI index.

Period-on-period inflation swaps are not as actively traded in the market as zero coupon inflation swaps although it is sometimes possible to get some prices. They are quoted by the fixed rate K for various maturities T_M. As explained earlier, we can use zero coupon inflation swaps to get real interest-rate discount factors. In principle, we could then use period-on-period inflation swaps (assuming we have enough of them) to calibrate the parameters (volatilities, mean reversion rates, CPI volatility, correlations) of the stochastic processes for the real interest-rate yield curve and the spot CPI index.

4.2 Zero coupon inflation swaps with delayed payment

In section 2.3, we valued standard zero coupon inflation swaps when the payment of the payoff of the swap occurred at the same time as the argument of the spot CPI index appearing in the numerator of the payoff. As we noted and explained in section 2.4, it is now relatively common to trade zero coupon inflation swaps where the payment is delayed for some time, perhaps several years or more. We refer to these inflation swaps as zero coupon inflation swaps with delayed payment. Our aim, in this section, is to derive a valuation formula for them. Unlike with a standard (i.e. with no delayed payment) zero coupon inflation swap, the valuation of zero coupon inflation swaps with delayed payment involves a convexity adjustment which is model-dependent.

Firstly, we derive a formula which, in a sense, extends the key observation of equation (2.21) to the situation of delayed payment, although we should stress that it is less general than equation (2.21), in the sense that it is no longer model-independent.

Proposition 1:

For any times t and T_N, with $t_0 \le t \le T_M \le T_N$, the following equation holds:

$$E_t\left[\exp\left(-\int_t^{T_N} r(s)\,ds\right)X(T_M)\right] = X(t)P_r(t,T_M)\frac{P(t,T_N)}{P(t,T_M)}\exp\left(\int_t^{T_M} C(s,T_M,T_N)ds\right) \tag{4.21}$$

where
$$\begin{aligned}\int_t^{T_M} C(s,T_M,T_N)ds &= \sum_{k=1}^{K_n}\sum_{j=1}^{K_r}\int_t^{T_M} \rho_{knjr}\left\{\sigma_{kn}(s,T_N)-\sigma_{kn}(s,T_M)\right\}\sigma_{jr}(s,T_M)ds \\ &+\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\int_t^{T_M} \rho_{knjn}\left\{\sigma_{kn}(s,T_M)-\sigma_{kn}(s,T_N)\right\}\sigma_{jn}(s,T_M)ds \\ &+\sum_{k=1}^{K_n}\int_t^{T_M} \rho_{knX}\sigma_X(s)\left\{\sigma_{kn}(s,T_N)-\sigma_{kn}(s,T_M)\right\}ds \end{aligned}\tag{4.22}$$

Proof: See Appendix 3.

Remark: When $T_M = T_N$, it is straightforward to verify that $C(s,T_M,T_N) \equiv 0$, in which case, equation (4.21) agrees with equation (2.21).

Remark: This formula will be used below to price zero coupon inflation swaps with delayed payment.

Suppose that today, time t_0, we enter into a zero coupon inflation swap with delayed payment. We denote the payment time of the payoff of the swap by T_N and we denote the maturity of the swap by T_M. We wish to value the swap, at time t, where $t_0 \le t \le T_M \le T_N$. The payoff of the zero coupon inflation swap with delayed payment is still:

$$N\left(\frac{X(T_M)}{X(t_0)}-1\right)-N\left((1+K)^{T_M}-1\right)$$

where K is the fixed rate on the swap and N is the notional amount.

But the payoff is paid at time T_N which is some time greater than or equal to T_M. The value, at time t, of the zero coupon inflation swap with delayed payment is :

$$\begin{aligned}
&E_t\left[\exp\left(-\int_t^{T_N} r(s)\,ds\right)\left(N\left(\frac{X(T_M)}{X(t_0)}-1\right)-N\left((1+K)^{T_M}-1\right)\right)\right]\\
&=E_t\left[N\exp\left(-\int_t^{T_N} r(s)\,ds\right)\left(\frac{X(T_M)}{X(t_0)}-(1+K)^{T_M}\right)\right]\\
&=\frac{N}{X(t_0)}E_t\left[\exp\left(-\int_t^{T_N} r(s)\,ds\right)X(T_M)\right]-NP(t,T_N)(1+K)^{T_M}\\
&=\frac{N}{X(t_0)}X(t)P_r(t,T_M)\frac{P(t,T_N)}{P(t,T_M)}\exp\left(\int_t^{T_M} C(s,T_M,T_N)ds\right)-NP(t,T_N)(1+K)^{T_M}
\end{aligned}$$

Remark: Notice that in the last line we have used proposition 1.

So the value of the swap, at time t, is:

$$\frac{N}{X(t_0)}X(t)P_r(t,T_M)\frac{P(t,T_N)}{P(t,T_M)}\exp\left(\int_t^{T_M} C(s,T_M,T_N)ds\right)-NP(t,T_N)(1+K)^{T_M} \qquad (4.23)$$

which gives us a valuation formula for the value of a zero coupon inflation swap with delayed payment at any time.

Remark: Comparing equation (4.23) with the equation for the value of a standard zero coupon inflation swap with no delayed payment (equation (2.32)), we can see that there is an extra term $\frac{P(t,T_N)}{P(t,T_M)}\exp\left(\int_t^{T_M} C(s,T_M,T_N)ds\right)$ in the inflation-linked leg.

4.3 Period-on-period inflation swaps with delayed payments

Our aim in this section is to value, at any time t, a period-on-period inflation swap with delayed payments.

The following proposition will be the key to this because it shows that when there are delayed payments, the valuation of period-on-period inflation swaps involves additional convexity adjustment terms. Equation (4.31) of proposition 2 extends equations (4.15) and (4.16) which we used in the valuation of period-on-period inflations swaps with no delayed payments.

Proposition 2:

When $t_0 \le t < T_{i-1} < T_i \le T_{N_i}$

$$E_t\left[\exp\left(-\int_t^{T_{N_i}} r(s)\,ds\right)\frac{X(T_i)}{X(T_{i-1})}\right] \tag{4.31}$$
$$= P(t,T_{i-1})\frac{P(t,T_{N_i})}{P(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\exp\left(\int_{T_{i-1}}^{T_i} C(s,T_i,T_{N_i})ds + \int_t^{T_{i-1}}\{A(s,T_{i-1},T_i)+B(s,T_{i-1},T_i,T_{N_i})\}ds\right)$$

where $C(s,T_i,T_{N_i})$ is given by (4.22), $A(s,T_{i-1},T_i)$ is given by (4.16) and

$$\begin{aligned}\int_t^{T_{i-1}} B(s,T_{i-1},T_i,T_{N_i})ds &= \sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\int_t^{T_{i-1}} \rho_{knjn}\left\{\sigma_{kn}(s,T_{i-1})-\sigma_{kn}(s,T_i)\right\}\sigma_{jn}(s,T_{N_i})ds \\ &+\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\int_t^{T_{i-1}} \rho_{knjn}\left\{\sigma_{kn}(s,T_i)-\sigma_{kn}(s,T_{i-1})\right\}\sigma_{jn}(s,T_i)ds \\ &+\sum_{k=1}^{K_n}\sum_{j=1}^{K_r}\int_t^{T_{i-1}} \rho_{knjr}\left\{\sigma_{jr}(s,T_i)-\sigma_{jr}(s,T_{i-1})\right\}\sigma_{kn}(s,T_{N_i})ds \\ &+\sum_{k=1}^{K_n}\sum_{j=1}^{K_r}\int_t^{T_{i-1}} \rho_{knjr}\left\{\sigma_{jr}(s,T_{i-1})-\sigma_{jr}(s,T_i)\right\}\sigma_{kn}(s,T_i)ds\end{aligned} \tag{4.32}$$

Proof: See Appendix 4.

Remark: Notice that when $T_i = T_{N_i}$, it is straightforward to confirm $B(s,T_{i-1},T_i,T_{N_i})$ and $C(s,T_i,T_{N_i})$ in equation (4.31) becomes zero, which confirms consistency with equation (4.15).

Suppose that today, time t_0, we enter into a period-on-period inflation swap with delayed payments. The swap is defined via a set of fixed dates $T_0 < T_1 < T_2 < ... < T_{M-1} < T_M$, where $T_0 \equiv t_0$.

The period-on-period inflation swap is made up of a series of swaplets. The key issue is that the value of the payoff of each swaplet is the same as the payoff of the corresponding swaplet of a period-on-period inflation swap with no delayed payments but now the payoff is paid at time T_{N_i} which is some time greater than or equal to T_i.

From equation (4.11), the payoff of the i^{th} swaplet, for $i = 1, 2, ..., M$, at time T_{N_i} is:

$$N\tau_{i,\inf}\left(\frac{X(T_i)}{X(T_{i-1})} - 1\right) - N\tau_{i,fixed}K$$

where the notation is the same as in equation (4.11).

The value, at time t, of the swaplet with delayed payment, i.e. $T_{N_i} \geq T_i$, is:

$$E_t\left[\exp\left(-\int_t^{T_{N_i}} r(s)\,ds\right)\left(N\tau_{i,\inf}\left(\frac{X(T_i)}{X(T_{i-1})} - 1\right) - N\tau_{i,fixed}K\right)\right]$$

$$= N\tau_{i,\inf}E_t\left[\exp\left(-\int_t^{T_{N_i}} r(s)\,ds\right)\frac{X(T_i)}{X(T_{i-1})}\right] - P\left(t, T_{N_i}\right)N\left(\tau_{i,\inf} + \tau_{i,fixed}K\right) \quad (4.33)$$

To value the floating (inflation-linked) side, we have to consider separately two different cases depending upon whether $t \geq T_{i-1}$ or $t < T_{i-1}$.

First case: $T_{i-1} \leq t \leq T_i$.

In this case, $X(T_{i-1})$ is known at time t. Therefore we can take $X(T_{i-1})$ outside of the expectation and write

$$E_t\left[\exp\left(-\int_t^{T_{N_i}} r(s)ds\right)\frac{X(T_i)}{X(T_{i-1})}\right]=\frac{1}{X(T_{i-1})}E_t\left[\exp\left(-\int_t^{T_{N_i}} r(s)ds\right)X(T_i)\right]$$
$$=\frac{1}{X(T_{i-1})}X(t)P_r(t,T_i)\frac{P\left(t,T_{N_i}\right)}{P\left(t,T_i\right)}\exp\left(\int_t^{T_i} C(s,T_i,T_{N_i})ds\right)$$

where we have used proposition 1 in the last line.

Hence, equation (4.33) becomes:

$$N\tau_{i,\inf}\frac{X(t)P_r(t,T_i)}{X(T_{i-1})}\frac{P\left(t,T_{N_i}\right)}{P\left(t,T_i\right)}\exp\left(\int_t^{T_i} C(s,T_i,T_{N_i})ds\right)-P\left(t,T_{N_i}\right)N\left(\tau_{i,\inf}+\tau_{i,fixed}K\right) \quad (4.34)$$

Second case: $t_0 \le t < T_{i-1}$.

Using the law of iterated expectations in equation (4.33) and by proposition 2, we can write the value of the swaplet, at time t, as :

$$N\tau_{i,\inf}P\left(t,T_{i-1}\right)\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\frac{P\left(t,T_{N_i}\right)}{P\left(t,T_i\right)}\exp\left(\int_{T_{i-1}}^{T_i} C(s,T_i,T_{N_i})ds+\int_t^{T_{i-1}}\{A\left(s,T_{i-1},T_i\right)+B(s,T_{i-1},T_i,T_{N_i})\}ds\right)$$
$$-P\left(t,T_{N_i}\right)N\left(\tau_{i,\inf}+\tau_{i,fixed}K\right) \quad (4.35)$$

Therefore, we can value a period-on-period inflation swap with delayed payments by summing up the value of all the swaplets, being careful to use equation (4.34) when $T_{i-1} \le t \le T_i$, and equation (4.35) when $t_0 \le t < T_{i-1}$.

5 An application of our convexity adjustment formulae to pricing LPI swaps

Limited price indexation (henceforth LPI) swaps are a type of exotic inflation derivative and are very common in the United Kingdom owing to the rules by which UK pension funds are governed. The rules often require that the future benefits of people paying into many pension funds have to rise by the year-on-year inflation rate whenever the year-on-year inflation rate, expressed as a percentage, is between some given levels $f\%$ and $c\%$, where $c \geq f$. If the year-on-year inflation rate is less than $f\%$, the future benefits have to increase by at least $f\%$ and if the year-on-year inflation rate is greater than $c\%$, the future benefits are increased by only $c\%$. In practice, f is often 0 % and c is often either 3 % or 5 % but variations do occur.

These rules effectively define the future liabilities of UK pension funds. Unsurprisingly, there has been substantial demand for inflation derivatives, from UK pension funds, which will give payoffs which can hedge against those liabilities. This has provided the economic rationale for LPI swaps.

We will see that we can use the convexity adjustment formulae, that we derived in chapter 4, to help price LPI swaps.

5.1 LPI swaps

Suppose that today, time t_0, we enter into an LPI swap. The LPI swap is defined via a set of fixed dates $T_0 < T_1 < T_2 < \cdots < T_{M-1} < T_M$, where $T_0 \equiv t_0$. In practice, these dates are usually approximately one year apart but they need not be. The payment of the payoff of the swap occurs at time T^*, where $T^* = T_M$. The payoff of the inflation-linked leg of the swap at time T^* is:

$$\prod_{i=1}^{M}\left(1+\min\left(\max\left(\frac{X(T_i)}{X(T_{i-1})}-1,F\right),C\right)\right) \text{ or equivalently } \prod_{i=1}^{M}\min\left(\max\left(\frac{X(T_i)}{X(T_{i-1})},1+F\right),1+C\right)$$

where C and F are constants, with $C \geq F$. In practice, F is often zero but we will assume in the following that C and F can take on any values (positive, negative or zero) provided $C \geq F$. The period-on-period rate of inflation between T_{i-1} and T_i is given by $X(T_i)/X(T_{i-1})-1$. So the role of the constants C and F is to cap and floor the period-on-period inflation rate over each period.

Remark: When $M=1$, LPI swaps could be priced by a variant of the Black (1976) formula. When $C=\infty$ and $F=-\infty$, the product telescopes and the LPI swap has the same payoff as a zero coupon inflation swap. However, when C and F are finite and $M>1$, we would need to price a swap whose payoff is path-dependent. Because of the path-dependency, they are not, in general, trivial to price. When $M=2$ or $M=3$ they could be priced by numerical integration techniques (i.e. quadrature for the case $M=2$ and cubature for the case $M=3$). However, in practice, LPI swaps typically have maturities anywhere between 5 years and 40 years implying that M is between 5 and 40. When $M \geq 4$, the only feasible methodology to precisely price LPI swaps is Monte Carlo simulation but this is CPU intensive. Hence, it would be desirable to have a fast, even if approximate, quasi-analytic methodology to price them. Such a methodology is proposed in Ryten (2007).

5.2 Pricing LPI swaps

In this section, we will use the methodology of Ryten (2007) to get an approximate pricing formula for the inflation-linked leg of LPI swaps. However, firstly, we introduce some notation: We denote by Q^{T^*} the probability measure defined with respect to the numeraire which is the zero coupon bond maturing at time T^*. We denote by $E_t^{T^*}[\ ]$ expectations, at time t, with respect to Q^{T^*}.

The methodology of Ryten (2007) is fully explained in Ryten (2007) so we will just outline the approach here. It uses the idea of common factor representation.

Suppose that we have a T_M year LPI swap with M periods.

Let X_i denote $\dfrac{X(T_i)}{X(T_{i-1})}$, for $i=1,2,\ldots,M$. We will show in Appendix 5 that:

- $\ln X_i \equiv \ln \dfrac{X(T_i)}{X(T_{i-1})}$, for each i, $i=1,2,\ldots,M$, is distributed as multi-variate normal in our model.

- We can calculate the covariance matrix $\operatorname{cov}\left(\ln X_i, \ln X_j\right)$ (5.21)

  for each i, j, $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, M$.

In general, none of the elements of this covariance matrix will be zero because $\ln X_i$ will not be independent of $\ln X_j$ for any i and j. This lack of independence severely complicates the problem of pricing an LPI swap. The key idea of Ryten (2007) (see also Jackel (2004)) is to replace the covariance matrix (5.21) for each i, j by another matrix, which is close to the actual correlation matrix in some sense, but in which the off-diagonal elements have a simple structure.

We use the same notation as in Ryten (2007). We write X_i in the form $X_i := \exp(a_i z_i + b_i)$ where $z_i \sim N(0,1)$; $\operatorname{cov}(\ln X_i, \ln X_j) = \operatorname{cov}(z_i, z_j) \cdot a_i \cdot a_j$; $E\left[X_i\right] = \exp(b_i + \frac{1}{2} a_i^2)$.

The key idea of Ryten (2007) is to replace X_i by $\hat{X}_i := \exp(b_i + a_i(\hat{a}_i w + \hat{d}_i \varepsilon_i))$, with the following additional properties: The system $w, \varepsilon_1, \ldots, \varepsilon_M$ is a system of independent $N(0,1)$ variates, and for each i, $\hat{a}_i^2 + \hat{d}_i^2 = 1$. Ryten (2007) shows how to calculate $\hat{a}_i$ and $\hat{d}_i$, for each i. The variates $\hat{X}_1, \ldots, \hat{X}_M$ are a representation of the variates $X_1, \ldots, X_M$ via one common factor w and additional individual idiosyncratic random variables ε_i, $i = 1, 2, \ldots, M$.

By changing measure to Q^{T^*} and using Girsanov's Theorem, the price, at time t_0, of the inflation-linked leg of the LPI swap is:

$$E_{t_0}\left[\exp\left(-\int_{t_0}^{T^*} r(s)ds\right)\prod_{i=1}^{M}\min\left(\max\left(\frac{X(T_i)}{X(T_{i-1})}, 1+F\right), 1+C\right)\right]$$
$$= P(t_0, T^*) E_{t_0}^{T^*}\left[\prod_{i=1}^{M}\min\left(\max\left(\frac{X(T_i)}{X(T_{i-1})}, 1+F\right), 1+C\right)\right]$$

$$\begin{aligned}
&\simeq P(t_0,T^*)E_{t_0}^{T^*}\left[\prod_{i=1}^{M}\min\left(\max\left(\hat{X}_i,1+F\right),1+C\right)\right]\\
&=P(t_0,T^*)E_{t_0}^{T^*}\left[E_{t_0}^{T^*}\left(\prod_{i=1}^{M}\min\left(\max\left(\hat{X}_i,1+F\right),1+C\right)\middle| w\right)\right]\\
&=P(t_0,T^*)E_{t_0}^{T^*}\left[\prod_{i=1}^{M}E_{t_0}^{T^*}\left(\min\left(\max\left(\hat{X}_i,1+F\right),1+C\right)\middle| w\right)\right] \qquad (5.22)
\end{aligned}$$

Remark: By assumption, the ε_i are independent, and consequently, conditional on a specific value of w, the variates $\hat{X}_i$ are independent, i.e. $\mathrm{cov}(\hat{X}_i \mid w, \hat{X}_j \mid w) = 0$, when $i \neq j$. Therefore, we see that the conditional expectation of the product in the last but one line of equation (5.22) becomes a product of conditional expectations in the last line. We have used $\simeq$ (approximately equals) in the third line of equation (5.22) because the variates $\hat{X}_i$ are, in general, only an approximate representation of the variates X_i, $i = 1, 2, \ldots, M$. When $M \leq 2$, the representation is exact. When $M \geq 3$, the representation is only approximate. It is true that $E_{t_0}^{T^*}[\hat{X}_i] = E_{t_0}^{T^*}[X_i]$, $\mathrm{var}_{t_0}[\ln \hat{X}_i] = \mathrm{var}_{t_0}[\ln X_i]$ for all i, for any value of M but when $M \geq 3$, then $\mathrm{cov}(\hat{X}_i, \hat{X}_j)$ is only an approximation to $\mathrm{cov}(X_i, X_j)$, when $i \neq j$.

We can use the methodology of Ryten (2007) to evaluate equation (5.22) provided that we can compute the expectation and variance of X_i in the probability measure Q^{T^*}. We do this in Appendix 5. Since X_i is lognormal (see Appendix 5), then denoting by $\mu_{\ln X_i}$ and $\sigma^2_{\ln X_i}$ the mean and variance of $\ln X_i$, then $E_{t_0}^{T^*}\left[X_i\right] = \exp(\mu_{\ln X_i} + \frac{1}{2}\sigma^2_{\ln X_i})$ for $i = 1, 2, \ldots, M$. Hence, we can get the expectation of $\ln X_i$, i.e. $\mu_{\ln X_i} = \ln\left(E_{t_0}^{T^*}\left[X_i\right]\right) - \frac{1}{2}\sigma^2_{\ln X_i}$.

Now we can use the following result:

If $X \sim N(\mu_X, \sigma_X^2)$, $W \sim N(0,1)$ *and* ρ_{XW} *is the correlation between* X *and* W, *then* $X \mid W = w$ *is normally distributed and, furthermore,*

$$E\left[X \middle| W=w\right]=\mu_X+\rho_{XW}\sigma_X w,\ \ Var\left[X \middle| W=w\right]=\sigma_X^2(1-\rho_{XW}^2)$$

In Appendix 5, we show that the correlation between $\ln \hat{X}_i$ and w is $\hat{a}_i$, for each i, $i=1,2,\ldots,M$.

Now, we recall that $E_{t_0}^{T^*}[\ln \hat{X}_i]=E_{t_0}^{T^*}[\ln X_i]=\mu_{\ln X_i}$, $\mathrm{var}_{t_0}[\ln \hat{X}_i]=\mathrm{var}_{t_0}[\ln X_i]=\sigma_{\ln X_i}^2$, then, using the result above, we get

$$E_{t_0}^{T^*}[\ln \hat{X}_i \mid w]=\mu_{\ln X_i}+\hat{a}_i\,\sigma_{\ln X_i}w,\ \ \overline{\sigma}_i^2 := var_{t_0}[\ln \hat{X}_i \mid w]=\sigma_{\ln X_i}^2(1-\hat{a}_i^2)$$

$$\overline{F}_i := E_{t_0}^{T^*}[\hat{X}_i \mid w]=\exp\left(\mu_{\ln X_i}+\hat{a}_i\,\sigma_{\ln X_i}w+\frac{1}{2}\overline{\sigma}_i^2\right), \qquad i=1,2,\ldots,M.$$

Finally (see Ryten (2007)), equation (5.22) becomes:

$$P(t_0,T^*)E_{t_0}^{T^*}\left[\prod_{i=1}^{M}\left(\overline{F}_i-Call(\overline{F}_i,1+C,\overline{\sigma}_i)+Put(\overline{F}_i,1+F,\overline{\sigma}_i)\right)\right] \tag{5.23}$$

where $Call(\overline{F}_i,1+C,\overline{\sigma}_i)$ and $Put(\overline{F}_i,1+F,\overline{\sigma}_i)$ are the undiscounted prices of a call option, with strike $1+C$, and a put option, with strike $1+F$, in the Black (1976) formula, when the forward price is $\overline{F}_i$ and the integrated variance is $\overline{\sigma}_i^2$.

Remark: Note that each term in the product in equation (5.23) depends on the common factor w, through $\overline{F}_i$ and $\overline{\sigma}_i$, and w has a standard normal $N(0,1)$ distribution.

We can then evaluate equation (5.23) by numerically integrating the product of $\prod_{i=1}^{M}\left(\overline{F}_i-Call(\overline{F}_i,1+C,\overline{\sigma}_i)+Put(\overline{F}_i,1+F,\overline{\sigma}_i)\right)$ and the standard normally density function. This gives us the price of the LPI swap at time t_0 (note that when $M \geq 3$, it is only an approximation).

6 Calibration to market data

We will now calibrate our model to market data in this section.

6.1 Get historical data

6.11 Get zero coupon inflation swaps and CPI

We obtained the fixed rates being quoted in the market for sterling (GBP) denominated zero coupon inflation swap rates, for every working day between 9th July 2003 and 14th June 2007, with maturities, equal to 5 years, 10 years, 15 years, 20 years, 25 years and 30 years. Using equation (2.33), this allowed us to get data for real zero coupon prices, for every working day, for these six maturities.

We also obtained historical data on the CPI index (recall that, throughout this paper, we call the index the CPI for brevity but, in actual fact, we used the UK RPI index). Because the CPI data is only available monthly, we decided to use only monthly data (even though we had daily data) for nominal and real zero coupon bond prices. We decided to use data for the 28th (or the closest working day) of each month.

Remark: Since CPI is announced in the middle of a month (i.e. 10th-20th), the data is somewhat noisy so we decided not to use data from this period. The choice is fairly arbitrary but we decided to use data from the 28th of each month (or the closest working day).

6.12 Get nominal and real discount factors

We obtained nominal discount factors, in sterling (GBP), for every working day between 1st July 2003 and 22nd June 2007, with maturities, again, equal to 5 years, 10 years, 15 years, 20 years, 25 years and 30 years. These were obtained, in the standard fashion, from GBP LIBOR deposit rates and by bootstrapping GBP swap rates.

Jarrow and Yildirim (2003) consider the valuation of inflation-linked instruments in the context of the market for Treasury Inflation Protected Securities (henceforth TIPS). TIPS are US Treasury bonds whose coupon and principal payments are linked to US CPI. Since these are coupon bearing bonds, Jarrow and Yildirim (2003) had to use a stripping methodology to extract the prices of real zero coupon bonds from the prices of coupon bearing bonds. They then used historical data of these real zero coupon bond prices to estimate the parameters of their model. By contrast, we are working within the context of inflation swaps. We know from equation (2.33) that there is a simple relationship between the fixed rate quoted on standard zero inflation swaps and real zero coupon bond prices. Hence, we do not need to employ the stripping algorithm of Jarrow and Yildirim (2003) in order to get real zero coupon bond prices. We simply used equation (2.33).

Before we describe how we estimated the model parameters, we need to be more precise about the specific form of the model that we used. In chapters 3, 4, and 5, we worked with a very general multi-factor version of the Jarrow and Yildirim (2003) model. However, we need to bear in mind that there is not too much historical data for inflation swaps and what data there is, may be somewhat noisy. Hence, in order to make for a simpler estimation of parameters, throughout this chapter, we assumed that real zero coupon bond prices are driven by just a single Brownian motion i.e. we assumed $K_r = 1$ in equation (3.21) to simplify parameter estimation. In addition, we assume that the volatility $\sigma_X(t)$ of the spot CPI index is constant, i.e. we assume that $\sigma_X(t) \equiv \sigma_X$. We considered two possible specifications for nominal zero coupon bond prices, namely that there is either one Brownian motion driving nominal zero coupon bond prices or that there are two i.e. either $K_n = 1$ or $K_n = 2$.

6.2 Get model parameters using Jarrow and Yildirim (2003) method

In order to price the exotic inflation derivatives we discussed in chapters 4 and 5, we need model parameters which are dependent on the specific model. Therefore, firstly, we need to specify the volatility functions $\sigma_{kn}(t,T)$ and $\sigma_{jr}(t,T)$.

Potentially, there are different forms of the volatility functions, $\sigma_{kn}(t,T)$,

$\sigma_{jr}(t,T)$ for each k, j, $k = 1,...,K_n$, $j = 1,...,K_r$, but we will consider only the extended Vasicek form, where for each k, j we assume

$$\sigma_{kn}(t,T) \equiv \frac{\sigma_{nk}}{\alpha_{nk}}\left(1-\exp\left(-\alpha_{kn}(T-t)\right)\right) \tag{6.21}$$

$$\sigma_{jr}(t,T) \equiv \frac{\sigma_{rj}}{\alpha_{rj}}\left(1-\exp\left(-\alpha_{jr}(T-t)\right)\right) \tag{6.22}$$

where, for each k, each j, σ_{nk}, α_{nk}, σ_{rj} and α_{rj} are positive constants.

6.21 One factor model

In this sub-section, we estimate model parameters for the case where we have one Brownian motion driving nominal zero coupon bond prices i.e. when $K_n = 1$. In addition as already stated, we assume $K_r = 1$. Using the method of Jarrow and Yildirim (2003), the variance of zero coupon bond prices over the time interval $[t, t+\triangle]$ satisfies the following equations:

$$\mathrm{var}\left(\frac{\triangle P_r(t,T)}{P_r(t,T)}\right) = \frac{\sigma_{r1}^2(1-e^{-\alpha_{r1}(T-t)})^2\triangle}{\alpha_{r1}^2} \tag{6.23}$$

$$\mathrm{var}\left(\frac{\triangle P_n(t,T)}{P_n(t,T)}\right) = \frac{\sigma_{n1}^2(1-e^{-\alpha_{n1}(T-t)})^2\triangle}{\alpha_{n1}^2} \tag{6.24}$$

Using Excel Solver, we ran a cross sectional non linear regression based on the equations (6.23) and (6.24) acoss the six different maturities to estimate the parameters $(\sigma_{n1}, \alpha_{n1})$ and $(\sigma_{r1}, \alpha_{r1})$. To be precise, in our calibration, we solved for the parameters which minimized the sums of squares of differences between the historical volatilities of zero coupon bond prices of the six different maturities and the model volatilities in equations (6.21) and (6.22). The historical volatilities were estimated using monthly data i.e. $\triangle = 1/12$.

The estimates of these parameters are $\sigma_{r1} = 0.006094, \alpha_{r1} = 0.032193, \sigma_{n1} = 0.007242$,

$\alpha_{n1} = 0.043585$ as given in table 3. These parameters provide the volatility inputs needed for the convexity adjustments (see expression (4.16), (4.22), (4.32))

In this sub-section, we only use a one factor model when estimating model parameters, which means we only need ρ_{krX}, ρ_{jnX}, ρ_{krjn} when $k=1$ and $j=1$. From the method of Jarrow and Yildirim (2003), these parameters have the following expression:

$$\begin{aligned} \sigma_X &= \left\{ \frac{1}{\triangle} \mathrm{var}\left(\frac{\triangle X(t)}{X(t)} \right) \right\}^{\frac{1}{2}} & \rho_{1rX} &= cor\left(\frac{\triangle P_{r1}(t,T)}{P_{r1}(t,T)}, \frac{\triangle X(t)}{X(t)} \right) \\ \rho_{1nX} &= cor\left(\frac{\triangle P_{n1}(t,T)}{P_{n1}(t,T)}, \frac{\triangle X(t)}{X(t)} \right) & \rho_{1n1r} &= cor\left(\frac{\triangle P_{n1}(t,T)}{P_{n1}(t,T)}, \frac{\triangle P_{r1}(t,T)}{P_{r1}(t,T)} \right) \end{aligned} \tag{6.25}$$

Remark: We use the historical data of CPI, nominal zero coupon bond prices and real zero coupon bond prices, as of the 28th of each month, to estimate these parameters. The parameters obtained are in tables 2 and 3.

6.22 Two factor model

For the case where we have two Brownian motions driving nominal zero coupon prices i.e. when $K_n = 2$, we obtained the parameters σ_{n1}, α_{n1}, σ_{n2}, α_{n2}, ρ_{n1n2} by calibrating a two-factor Gaussian HJM model (Heath et al (1992), Babbs (1990), Hull and White (1993)) model to the market prices of liquid European swaptions. The results of the calibration were that we obtained model parameters as follows:

$$\sigma_{n1} = 0.006498, \alpha_{n1} = 0.064945, \sigma_{n2} = 0.006332, \alpha_{n2} = 0.000016$$

together with the correlation between these two factors $\rho_{n1n2} = -0.462963$ as given in tables 4 and 5. These parameters were provided by John Crosby and Lloyds TSB.

We assumed that $\sigma_{r1}, \alpha_{r1}, \rho_{1rX}$ were as above (see also table 3). We also need to estimate $\rho_{1nX}, \rho_{2nX}, \rho_{1n1r}$ and ρ_{2n1r}. We used the approach described on page 425 of Brigo and Mercurio (2001). In essence, we set $\rho_{1nX} \equiv \rho_{2nX}$ and we set $\rho_{1n1r} \equiv \rho_{2n1r}$. We assume that ρ_{1nX} and ρ_{2nX} are equal to the corresponding values we obtained for the one factor case above (see table 2). That is, we assume $\rho_{1nX} \equiv \rho_{2nX} = 0.018398$. We would like to do likewise with ρ_{1n1r} and ρ_{2n1r}. However, if we set $\rho_{1n1r} \equiv \rho_{2n1r} = 0.7504$, we find that the correlation matrix is not positive definite. Therefore, we decided to set $\rho_{1n1r} \equiv \rho_{2n1r} = 0.5181$ because 0.5181 is the closest value which makes the correlation matrix positive definite.

Whilst we concede that this is unlikely to yield anything like perfect estimates of these parameters, there is (as Brigo and Mercurio (2001) explain) at least a measure of mathematical consistency about it and, in addition, given the relatively scarce amount of data for inflation, it is a pragmatic simplification.

6.3 Give the values of convexity adjustments of exotic derivatives

We have now obtained estimates of the model parameters needed for the convexity adjustments (see equations (4.16), (4.22), (4.32)). We will, later in this section, use these parameters to test the analytical formulae we derived in sections 4.1, 4.2 and 4.3, and then give some numerical examples and comparisons of the convexity adjustments, for the three types of inflation swaps we considered in chapter 4, for different swap tenors and payment times.

John Crosby (my industry supervisor) also provided data which gives the values of the convexity adjustment (together with standard errors of these estimates) using a Monte Carlo methodology which was used to test and benchmark the analytical formulae (equations (4.16), (4.22), (4.32)). The Monte Carlo simulation simulated the CPI index level and the nominal and real yield curves by simulating underlying Gaussian state variables and it therefore had no discretisation error bias. For the sake of brevity, we omit the full details since they can be found in, for example, Crosby (2005), Crosby (2007a), Dempster and Hutton (1997) and Glasserman (2004). The Monte Carlo values we report were computed using 130 million runs (65 million runs plus 65 million antithetic runs) which took several hours of CPU time.

Here, thanks again John Crosby for his kindly help.

In examples 1 to 4, we use the model parameters for the case when there is one Brownian motion driving nominal interest-rates i.e. when $K_n = 1$. We plot the convexity adjustments, graphically, in the different examples below.

Example 1: Comparison of the Monte Carlo results and the analytical formulae

In this example (figures 1, 2, and 3), we consider the convexity adjustments for zero coupon inflation swaps with delayed payment, period-on-period swaplets with no delayed payment and period-on-period swaplets with a 5 year payment delay.

Figure 1

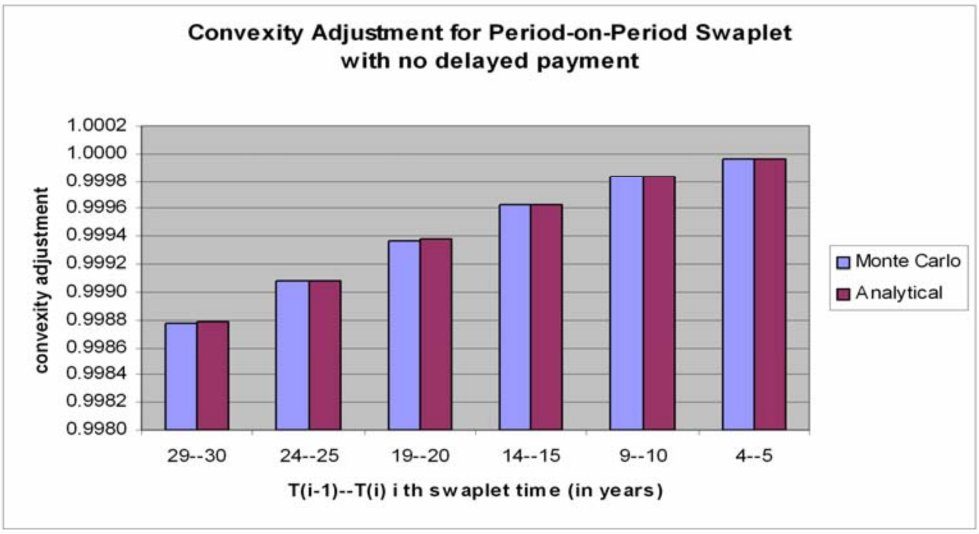

Figure 2

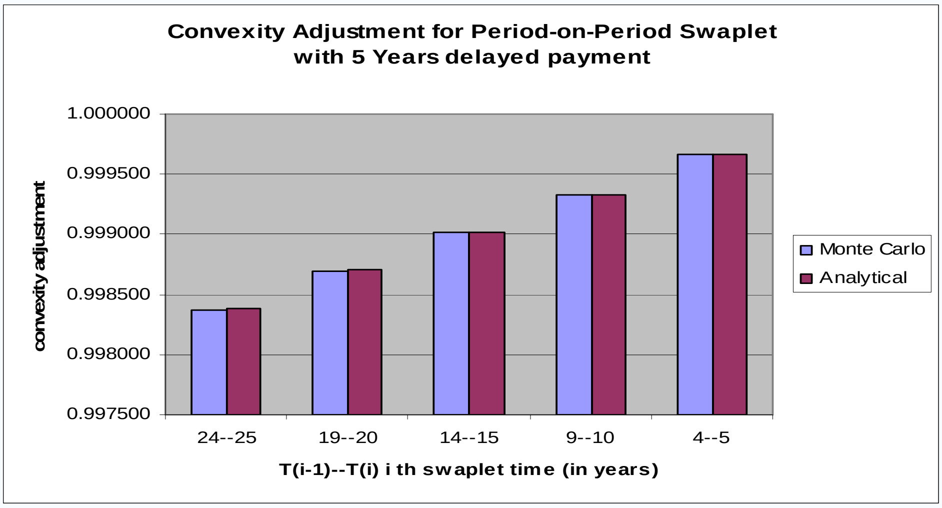

Figure 3

From figure 1, 2 and 3, we can see that the difference between the results from the Monte Carlo and the analytical results of chapter 4 are very small - in fact, they are almost zero and, whilst we have not displayed the standard errors in the graphs, we can confirm that the analytical results are consistent with the standard errors of the Monte Carlo simulation. We can conclude that the formulae we derived in chapter 4 are correct and that they have been correctly implemented.

Example 2: Convexity adjustments for zero coupon inflation swaps

In this example (figure 4), we compare the convexity adjustments for zero coupon inflation swaps, with maturities T_M equal to 25, 20, 15, 10 and 5 years, when there is no delayed payment, when there is a one year payment delay and when there is a five year payment delay.

From figure 4, we can see that, firstly, when there is no delayed payment time, the convexity adjustment always equals one, which is what we expect. However, when the payment delay is increased, from zero to one year to 5 years delay, the convexity adjustments get further away from one. In addition, as the maturity increases from 5 years to 25 years, the convexity adjustments also get further away from one. This illustrates that the convexity adjustments become more significant for longer maturities and longer payment delays.

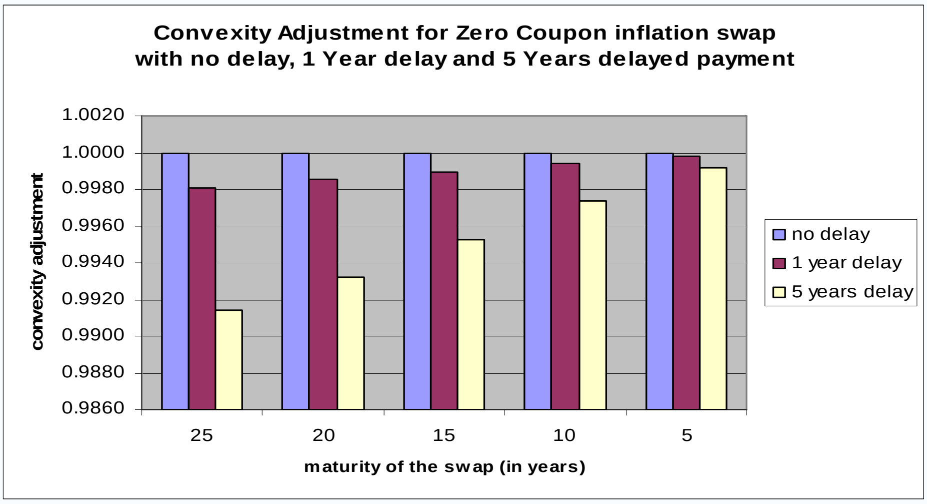

Figure 4

Example 3: Convexity adjustments for period-on-period swaplets

In this example (figure 5), we perform a similar analysis to example 2, but this time for period-on-period swaplets.

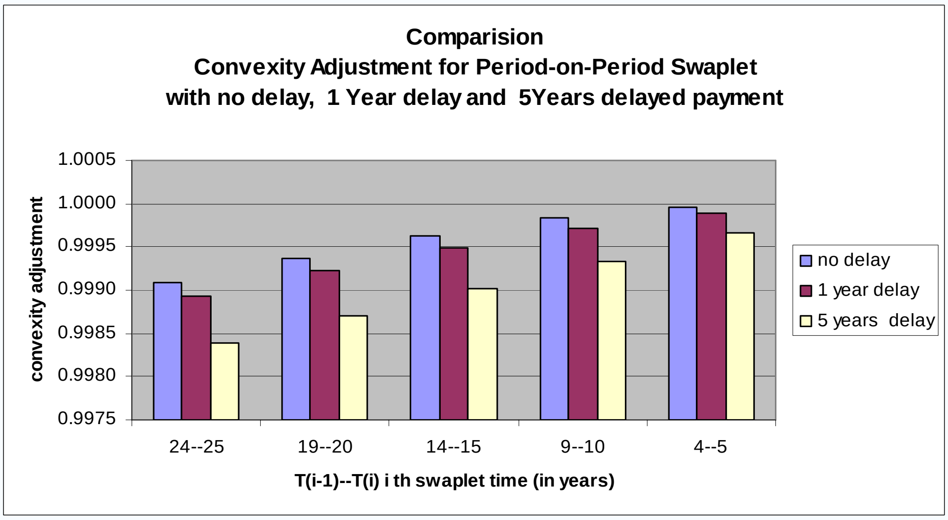

Figure 5

In figure 5, we again see that longer maturities and longer payment delays produce convexity adjustments which are further away from one.

Example 4: The effect of the convexity adjustment on the fixed rate for zero coupon inflation swaps.

Figure 6 shows the fixed rate K on zero coupon inflation swaps, with a payment delay of 5 years, for swaps of different tenors from 25 years to 5 years. The fixed rate on the swaps when we evaluate the convexity adjustment, using equation (4.22) and the parameters for the one factor case (see tables 2 and 3), is always lower than the fixed rate we would obtain on the swaps if we naively assumed that no convexity adjustment was necessary. Furthermore, the difference increases with increasing swap tenor. At 25 years, the difference is more than 0.035% which is, from a trader's perspective, significant as the bid-offer spread in the market, for zero coupon inflation swaps, is approximately 0.03%, or sometimes even less.

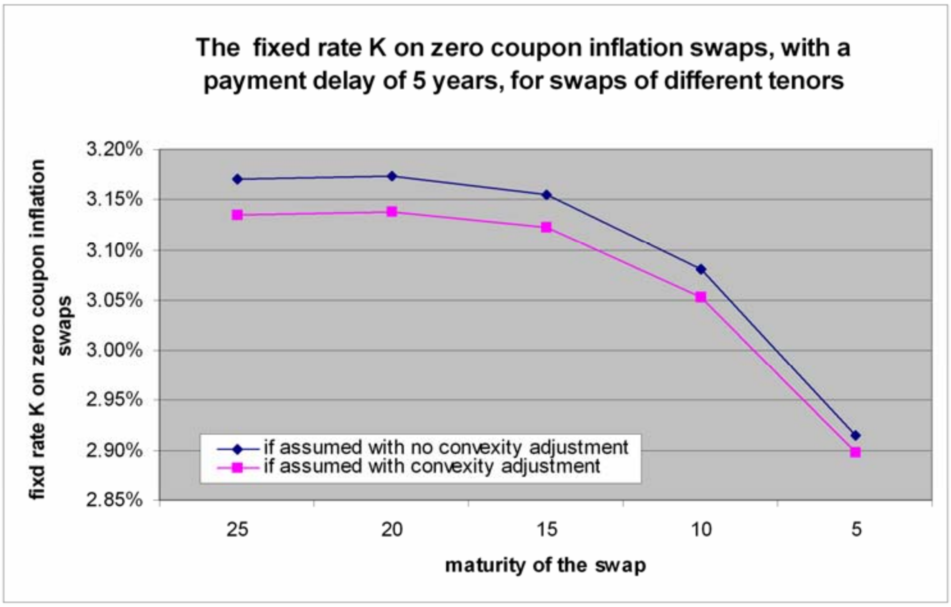

Figure 6

Example 5:

In example 5, we use the model parameters (see tables 4 and 5), for the case when there are two Brownian motions driving nominal interest-rates i.e. $K_n = 2$. We compare the estimates of the convexity adjustments, obtained by Monte Carlo simulation (we also report the standard errors in the column marked s/e) and those obtained using the analytical formulae, for period-on-period swaplets when there is no delayed payment, when there is a one year payment delay and when there is a five year payment delay.

The table shows again that the formulae we derived in chapter 4 are correct and that they have been correctly implemented.

Convexity Adjustment for period-on-period swaplet with delayed payment						
s/e	Monte Carlo	Analytical	difference	T_{N_i}	T_{i-1}	T_i
0.0000496	1.0008781	1.0008662	0.0000120	30	29	30
0.0000369	1.0006020	1.0006049	-0.0000028	25	24	25
0.0000262	1.0003785	1.0003860	-0.0000075	20	19	20
0.0000171	1.0002049	1.0002131	-0.0000081	15	14	15
0.0000096	1.0000872	1.0000882	-0.0000010	10	9	10
0.0000036	1.0000143	1.0000146	-0.0000003	5	4	5
0.0000392	1.0003033	1.0003045	-0.0000012	26	24	25
0.0000282	1.0001849	1.0001924	-0.0000076	21	19	20
0.0000189	1.0000951	1.0001040	-0.0000089	16	14	15
0.0000110	1.0000361	1.0000386	-0.0000025	11	9	10
0.0000046	0.9999993	1.0000003	-0.0000010	6	4	5
0.0000497	0.9990488	0.9990379	0.0000109	30	24	25
0.0000370	0.9993626	0.9993657	-0.0000031	25	19	20
0.0000263	0.9996226	0.9996306	-0.0000081	20	14	15
0.0000172	0.9998125	0.9998204	-0.0000079	15	9	10
0.0000097	0.9999332	0.9999359	-0.0000027	10	4	5

Table 6

Now, in examples 6 and 7, we will give some examples of the prices of LPI swaps. We use the one factor model parameters (see tables 2 and 3). For the purposes of these illustrations, we assumed that the interest-rate (both nominal and real) yield curves were initially flat and that nominal interest rates to all maturities were 0.05 and real interest rates to all maturities were 0.025 i.e. we assumed $P(t_0,T)\equiv\exp(-0.05T)$ and $P_r(t_0,T)\equiv\exp(-0.025T)$ for all T.

Example 6:

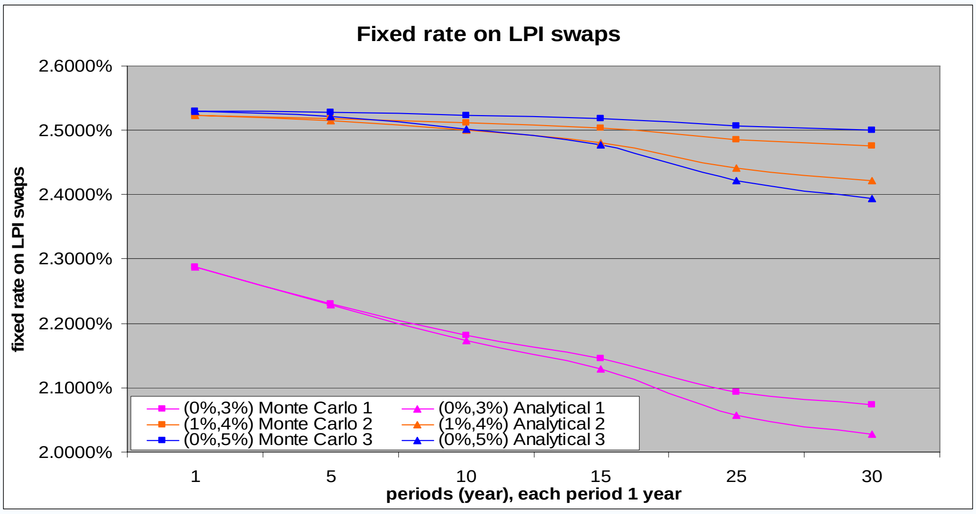

Figure 7

In figures 7, we consider three different combinations of floors and caps (which are commonly traded in the market) namely, (0%, 3%), (0%, 5%) and (1%, 4%). For all three different combinations, we considered LPI swaps where each period was equal to one year, but the number of periods varied from one period, through 5, 10, 15, 25 to 30 periods and hence the maturities of the LPI swaps varied from one year to 30 years. We can see that the fixed rates obtained from the quasi-analytical methodology of Ryten (2007) (see chapter 5) are very close to the results obtained from Monte Carlo simulation for shorter maturities although the differences do increase for LPI swaps with longer maturities.

Example 7:

In this example, we considered eleven different combinations of floors and caps as indicated in table 7. We considered LPI swaps whose maturities were one year, six years, 10 years and 25 years. For all the swaps, except those with six year maturities, each period was a year and hence the number of periods equaled the number of years to maturity. By contrast, the LPI swaps with six year maturities had only two periods as each period was equal to 3 years.

1 year , 1 period LPI swap, each period 1 year									
cap	floor	s/e	Monte Carlo	Ryten(2007)price	difference	rate(MC) %	rate %	diff rates %
3.00%	0.00%	0.00000070	0.97299233	0.97299197	0.00000035	2.28787090	2.28783385	0.00003705
3.00%	2.00%	0.00000048	0.97509614	0.97509606	0.00000008	2.50903861	2.50903004	0.00000857
3.20%	1.00%	0.00000069	0.97392088	0.97392074	0.00000014	2.38548720	2.38547232	0.00001488
3.50%	0.50%	0.00000078	0.97431799	0.97431786	0.00000014	2.42723467	2.42722019	0.00001448
4.00%	1.00%	0.00000082	0.97523281	0.97523266	0.00000015	2.52340655	2.52339127	0.00001528
4.50%	1.75%	0.00000075	0.97662115	0.97662107	0.00000009	2.66935897	2.66934980	0.00000916
4.75%	0.25%	0.00000091	0.97528576	0.97528552	0.00000024	2.52897271	2.52894779	0.00002492
5.00%	0.00%	0.00000093	0.97529491	0.97529467	0.00000023	2.52993472	2.52991019	0.00002452
5.00%	0.50%	0.00000091	0.97538863	0.97538848	0.00000015	2.53978696	2.53977123	0.00001573
6.00%	0.00%	0.00000094	0.97534355	0.97534338	0.00000017	2.53504824	2.53503084	0.00001740
12.00%	-8.00%	0.00000095	0.97531015	0.97530991	0.00000023	2.53153668	2.53151205	0.00002463

6 year , 2 period LPI swap, each period 3 year								
cap	floor	s/e	Monte Carlo	Ryten(2007)price	difference	rate(MC) %	rate %	diff rates %
3.00%	0.00%	0.00003780	0.78495325	0.78495282	0.00000043	0.96914751	0.96913828	0.00000923
3.00%	2.00%	0.00000381	0.78535665	0.78535619	0.00000046	0.97779400	0.97778415	0.00000986
3.20%	1.00%	0.00000380	0.78796095	0.78796046	0.00000049	1.03352518	1.03351471	0.00001047
3.50%	0.50%	0.00000380	0.79217311	0.79217263	0.00000048	1.12334037	1.12333018	0.00001020
4.00%	1.00%	0.00000382	0.79927611	0.79927565	0.00000046	1.27389887	1.27388905	0.00000982
4.50%	1.75%	0.00000383	0.80629929	0.80629889	0.00000040	1.42167313	1.42166468	0.00000845
4.75%	0.25%	0.00000382	0.80933161	0.80933127	0.00000034	1.48514452	1.48513733	0.00000719
5.00%	0.00%	0.00000382	0.81256518	0.81256493	0.00000025	1.55261058	1.55260538	0.00000520
5.00%	0.50%	0.00000382	0.81261559	0.81261532	0.00000028	1.55366066	1.55365489	0.00000577
6.00%	0.00%	0.00000382	0.82473433	0.82473452	-0.00000020	1.80452207	1.80452609	-0.00000402
12.00%	-8.00%	0.00000408	0.85872395	0.85872618	-0.00000224	2.49208301	2.49212755	-0.00004453

10 year , 10 period LPI swap, each period 1 year								
cap	floor	s/e	Monte Carlo	Ryten(2007)price	difference	rate(MC) %	rate %	diff rates %
3.00%	0.00%	0.00000800	0.75265947	0.75199621	0.00066326	2.18204192	2.17303388	0.00900804
3.00%	2.00%	0.00000843	0.77648598	0.77635075	0.00013523	2.50099618	2.49921085	0.00178532
3.20%	1.00%	0.00000811	0.76363142	0.76309775	0.00053367	2.33003036	2.32287666	0.00715369
3.50%	0.50%	0.00000820	0.76599835	0.76515640	0.00084195	2.36170410	2.35044742	0.01125668
4.00%	1.00%	0.00000824	0.77730857	0.77638643	0.00092214	2.51184964	2.49968194	0.01216770
4.50%	1.75%	0.00000841	0.79369140	0.79295221	0.00073919	2.72588499	2.71631381	0.00957118
4.75%	0.25%	0.00000837	0.77800100	0.77653877	0.00146223	2.52097785	2.50169298	0.01928486
5.00%	0.00%	0.00000842	0.77818179	0.77659418	0.00158761	2.52335994	2.50242433	0.02093562
5.00%	0.50%	0.00000839	0.78021998	0.77876563	0.00145435	2.55018087	2.53104918	0.01913169
6.00%	0.00%	0.00000851	0.78000321	0.77827080	0.00173241	2.54733133	2.52453252	0.02279881
12.00%	-8.00%	0.00000857	0.77878970	0.77685651	0.00193319	2.53136615	2.50588632	0.02547983

25 year , 25 period LPI swap, each period 1 year								
cap	floor	s/e	Monte Carlo	Ryten(2007)price	difference	rate(MC) %	rate %	diff rates %
3.00%	0.00%	0.00001740	0.48090071	0.47664987	0.00425084	2.09322918	2.05697774	0.03625144
3.00%	2.00%	0.00001970	0.52903359	0.52824283	0.00079076	2.48352477	2.47739297	0.00613180
3.20%	1.00%	0.00001820	0.50313584	0.49993635	0.00319949	2.27797812	2.25188259	0.02609553
3.50%	0.50%	0.00001826	0.50586740	0.50064100	0.00522640	2.30013144	2.25764359	0.04248785
4.00%	1.00%	0.00001889	0.52928907	0.52356405	0.00572502	2.48550397	2.44093102	0.04457296
4.50%	1.75%	0.00002039	0.56353709	0.55875588	0.00478121	2.74285372	2.70784292	0.03501080
4.75%	0.25%	0.00001933	0.53128323	0.52153702	0.00974620	2.50092114	2.42503709	0.07588405
5.00%	0.00%	0.00001920	0.53195745	0.52111839	0.01083906	2.50612108	2.42174718	0.08437390
5.00%	0.50%	0.00001950	0.53709704	0.52734693	0.00975011	2.54555372	2.47043529	0.07511843
6.00%	0.00%	0.00001960	0.53823293	0.52584444	0.01238849	2.55421973	2.45874113	0.09547860
12.00%	-8.00%	0.00001985	0.53523622	0.52016130	0.01507492	2.53131892	2.41421620	0.11710272

Table 7

We can see that there is (to the probabilistic errors implied by the standard errors) perfect agreement between the prices of the LPI swaps obtained by Monte Carlo simulation and those obtained by the quasi-analytical methodology of Ryten (2007) (see chapter 5), for the LPI swaps with one year maturity (one period) and those with six years maturity (two periods of three years each). This is not surprising since we know that the quasi-analytical methodology is exact for the cases when $M \leq 2$. However, we see for the LPI swaps with 10 years maturity and 25 years maturity, the level of approximation involved in the quasi-analytical methodology. As a rough guide, the bid-offer spread in the market for LPI swaps is approximately 0.06% (expressed as the fixed rate on the swap). For the LPI swaps with 10 years maturity, the maximum (absolute) difference in the fixed rate, implied by the Monte Carlo results and the quasi-analytical methodology, is less than 0.026% which implies, if not perfect, certainly very accurate pricing as it is less than half the bid-offer spread. For the LPI swaps with 25 years maturity, the accuracy does deteriorate somewhat and is, in some cases, greater than the bid-offer spread in the market.

We also see that the accuracy of the quasi-analytical methodology, when $M \geq 3$, also deteriorates when the cap level is high and the floor level is low. This might initially seem surprising since in the limiting case that $C = \infty$ and $F = -\infty$, LPI swaps become the same as standard zero coupon swaps. However, the reason for the deterioration in accuracy is that the quasi-analytical methodology approximates the correlation structure.

Although, (using the notation of section 5.2), it is true that $E_{t_0}^{T^*}[\hat{X}_i] = E_{t_0}^{T^*}[X_i]$, for all i, and it is true that $E_{t_0}^{T^*}\left[\prod_{i=1}^{M} X_i\right] = E_{t_0}^{T^*}\left[\frac{X(T_M)}{X(t_0)}\right] = P_r(t_0, T_M) = P_r(t_0, T^*)$, the price of a standard zero coupon swap, the approximation of the correlation structure means that $E_{t_0}^{T^*}\left[\prod_{i=1}^{M} \hat{X}_i\right]$ does NOT equal $E_{t_0}^{T^*}\left[\prod_{i=1}^{M} X_i\right]$, when $M \geq 3$.

For the sake of brevity, we only considered the Ryten (2007) methodology for the case of conditioning on one common factor. Ryten (2007) also considers the case of conditioning on two common factors (which means evaluating the price of a LPI swap requires a double numerical integration). Ryten (2007) shows that (unsuprisingly) conditioning on two common factors gives a significant improvement in the accuracy of the methodology compared to using one common factor. We would certainly conjecture that using two common factors would also significantly improve the accuracy of the prices of the LPI swaps, with 10 years maturity and 25 years maturity, which we reported in table 7. However, we leave proof of this for future research.

7 Conclusions

The most actively traded inflation derivatives are standard (i.e. with no delayed payment) zero coupon inflation swaps. We have shown how these can be valued in a model-independent fashion and how they can be used to extract the term structure of real discount factors.

Recently, there has been a substantial increase in the demand for more exotic inflation derivative products. We have used a multi-factor version of the Jarrow and Yildirim (2003) model, which in turn is a Gaussian HJM (Heath et al. (1992)) model, to value some exotic inflation derivatives. The Jarrow and Yildirim (2003) model is based on the foreign exchange analogy which treats real zero coupon bond prices analogously to foreign (ie denominated in foreign currency) zero coupon bond prices while the CPI index which links the nominal and real economies plays the analogous role as the spot foreign exchange rate which links the domestic and foreign currencies.

Using the multi-factor Jarrow and Yildirim (2003) model, we have valued zero coupon inflation swaps with delayed payment, period-on-period inflation swaps with no delayed payments and period-on-period inflation swaps with delayed payments.

We have particularly focused on the convexity adjustments which arise in the valuation of these latter products, including those convexity adjustments which arise from the delay in the payment of the payoff of the swap in question.

Just as with using a Gaussian HJM (Heath et al. (1992)) model to price some exotic interest-rate derivatives, we can conclude that a major advantage of using the Jarrow and Yildirim (2003) model is that it is possible to price some exotic inflation derivatives with exact analytical formulae rather than with ad-hoc methodologies or time-consuming numerical methods.

Appendix 1

We will make frequent use of the following equations

$$\exp\left(-\int_t^{T_N} r(s)\,ds\right) = P(t,T_N)\exp\left(-\int_t^{T_N}\frac{1}{2}\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{knjn}\sigma_{kn}(s,T_N)\sigma_{jn}(s,T_N)ds\right) \exp\left(\int_t^{T_N}\sum_{k=1}^{K_n}\sigma_{kn}(s,T_N)dz_{kn}(s)\right) \tag{1}$$

$$\frac{P(T_{i-1},T_N)}{P(T_{i-1},T_i)} = \frac{P(t,T_N)}{P(t,T_i)}\exp\left(\int_t^{T_{i-1}}\sum_{k=1}^{K_n}\{\sigma_{kn}(s,T_N)-\sigma_{kn}(s,T_i)\}dz_{kn}(s)\right) \exp\left(\int_t^{T_{i-1}}\frac{1}{2}\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{knjn}\{\sigma_{kn}(s,T_i)\sigma_{jn}(s,T_i)-\sigma_{kn}(s,T_N)\sigma_{jn}(s,T_N)\}ds\right) \tag{2}$$

$$P_r(T_{i-1},T_i) = \frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\exp\left(\int_t^{T_{i-1}}\sum_{k=1}^{K_r}\{\sigma_{kr}(s,T_i)-\sigma_{kr}(s,T_{i-1})\}dz_{kr}(s)\right) \exp\left(\int_t^{T_{i-1}}\sum_{k=1}^{K_r}\rho_{krX}\sigma_X(s)\{\sigma_{kr}(s,T_{i-1})-\sigma_{kr}(s,T_i)\}ds\right) \exp\left(\int_t^{T_{i-1}}\frac{1}{2}\sum_{k=1}^{K_r}\sum_{j=1}^{K_r}\rho_{krjr}\{\sigma_{kr}(s,T_{i-1})\sigma_{jr}(s,T_{i-1})-\sigma_{kr}(s,T_i)\sigma_{jr}(s,T_i)\}ds\right) \tag{3}$$

Proof of equation (1): We can apply Ito's lemma to equation (3.11) to get an SDE for $\ln P(t,T_N)$ and then rewrite this equation in integral form from t to T_N, to express $P(T_N,T_N)$ in terms of $P(t,T_N)$ and $r(s)$. But then we note that $P(T_N,T_N)=1$. Rearranging, we get equation (1). As an aside, it is straightforward to confirm that:

$$E_t\left[\exp\left(-\int_t^{T_N} r(s)\,ds\right)\right] = P(t,T_N) \tag{4}$$

Proof of equation (2): We can, as in the proof of equation (1), solve the SDE of equation (3.11) to get, firstly, an equation for $P(T_{i-1},T_N)$ in terms of $P(t,T_N)$ and, secondly, an equation for $P(T_{i-1},T_i)$ in terms of $P(t,T_i)$. If we divide the first equation by the second, the terms involving the nominal short rate cancel and we obtain equation (2).

Proof of equation (3): We now solve the SDE of equation (3.21) to get, firstly, an equation for $P_r\left(T_{i-1},T_i\right)$ in terms of $P_r\left(t,T_i\right)$, and secondly, an equation for $P_r\left(T_{i-1},T_{i-1}\right)$ in terms of $P_r\left(t,T_{i-1}\right)$. If we divide the first equation by the second, and note that $P_r\left(T_{i-1},T_{i-1}\right)=1$, then the terms involving the real short rate cancel and we obtain equation (3).

Appendix 2

Consider a forward contract with maturity T_M. The payoff of this forward contract is $X(T_M)-K$, at time T_M. It costs nothing to enter into a forward contract and hence we choose K such that the forward contract has zero initial value, and the forward price is defined to be this value of K.

The price of the forward contract at time t is:

$$E_t\left[\exp\left(-\int_t^{T_M} r\left(s\right)ds\right)\left\{X\left(T_M\right)-K\right\}\right]=E_t\left[\exp\left(-\int_t^{T_M} r\left(s\right)ds\right)X\left(T_M\right)\right]-KP\left(t,T_M\right) \quad (1)$$

We choose K such that equation (1) equal zero. Solving the above equation, we get $K=E_t\left[\exp\left(-\int_t^{T_M} r\left(s\right)ds\right)X\left(T_M\right)\right]\Big/P\left(t,T_M\right)$.Hence, by definition, the forward price is:

$$F_X(t,T_M)=E_t\left[\exp\left(-\int_t^{T_M} r\left(s\right)ds\right)X\left(T_M\right)\right]\Big/P\left(t,T_M\right) \quad (2)$$

However, we can also show that in the absence of arbitrage $F_X\left(t,T_M\right)=X(t)P_r\left(t,T_M\right)/P\left(t,T_M\right)$.

To see this, consider two portfolio's A and B.

In portfolio A, we buy one forward contract, at time t. The payoff of the forward contract, at time T_M, is $X(T_M) - K$. K is chosen so that the value of the forward contract, at time t, is zero.

In portfolio B, at time t, we sell short K nominal bonds, for which we receive $KP(t, T_M)$ units of nominal currency, which by exchanging for real currency, gives us $KP(t, T_M) / X(t)$ units of real currency, which we then use to buy $KP(t, T_M) / X(t) P_r(t, T_M)$ notional amount of real bonds. The cost of portfolio B, at time t, is zero.

At time T_M, from the maturing real bonds, we receive $KP(t, T_M) / X(t) P_r(t, T_M)$ units of real currency which we sell at rate $X(T_M)$, to give us $\left(KP(t, T_M) / X(t) P_r(t, T_M)\right) \times X(T_M)$ units of nominal currency. We must also pay K units of nominal currency to repay the maturing nominal bonds. Therefore, the value of portfolio B at time T_M is $\left(KP(t, T_M) / X(t) P_r(t, T_M)\right) \times X(T_M) - K$.

We are free to choose K, however we wish. If we choose K to be $K = X(t) P_r(t, T_M) / P(t, T_M)$, then we can write the value of portfolio B, at time T_M, in the form $X(T_M) - K$.

But with this choice of K, portfolio B has the same value, at time T_M, as portfolio A and hence, in the absence of arbitrage, portfolio A must have the same value as portfolio B, at time t, but we know the latter has zero value. Hence, with the choice $K = X(t) P_r(t, T_M) / P(t, T_M)$, the forward contract has zero initial value.

Hence, by definition the forward price is: $$\frac{X(t) P_r(t, T_M)}{P(t, T_M)} = F_X(t, T_M) \tag{3}$$

From (2), (3), we have shown that $E_t\left[\exp\left(-\int_t^{T_M} r(s)ds\right)X(T_M)\right] = X(t)P_r(t,T_M)$

Appendix 3 Proof of Proposition 1

Proof:

Our aim is to compute the expectation in equation (4.21) of proposition 1. The computation is complicated by the fact that we have the stochastic discounting term $\exp\left(-\int_t^{T_N} r(s)ds\right)$ and we have the term $X(T_M)$ which has a stochastic drift. The key to computing the expectation will be to replace the stochastic discounting term $\exp\left(-\int_t^{T_N} r(s)ds\right)$ using equation (1) of Appendix 1 and to replace $X(T_M)$ by expressing it in terms of the forward CPI index $F_X(T_M)$. Then we will have simplified the expectation to computing the expectation of the product of log-normally distributed random variables.

Set $t = T = T_M$ in equation (3.32), i.e. $F_X(T_M,T_M) = \dfrac{X(T_M)P_r(T_M,T_M)}{P(T_M,T_M)} = X(T_M)$,

then $E_t\left[\exp\left(-\int_t^{T_N} r(s)ds\right)X(T_M)\right] = E_t\left[\exp\left(-\int_t^{T_N} r(s)ds\right)F_X(T_M,T_M)\right]$ and then from the form of equation (3.34):

$$
\begin{aligned}
&F_X(T_M,T_M) \qquad\qquad (4)\\
&= F_X(t,T_M)\exp\left(\int_t^{T_M}\frac{1}{2}\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{jnkn}\sigma_{jn}(s,T_M)\sigma_{kn}(s,T_M)ds - \int_t^{T_M}\frac{1}{2}\sigma_X^2(s)ds\right)\\
&\qquad \exp\left(-\int_t^{T_M}\frac{1}{2}\sum_{k=1}^{K_r}\sum_{j=1}^{K_r}\rho_{krjr}\sigma_{jr}(s,T_M)\sigma_{kr}(s,T_M)ds - \int_t^{T_M}\sum_{k=1}^{K_r}\rho_{krX}\sigma_X(s)\sigma_{kr}(s,T_M)ds\right)\\
&\qquad \exp\left(\int_t^{T_M}\left(\sigma_X(s)dz_X(s) + \sum_{k=1}^{K_r}\sigma_{kr}(s,T_M)dz_{kr}(s) - \sum_{k=1}^{K_n}\sigma_{kn}(s,T_M)dz_{kn}(s)\right)\right)
\end{aligned}
$$

We then substitute from equation (4) and from equation (1) of Appendix 1, and then we have:

$$E_t\left[\exp\left(-\int_t^{T_N} r(s)\,ds\right)F_X(T_M,T_M)\right] \tag{5}$$

$$=P(t,T_N)F_X(t,T_M)\exp\left(-\int_t^{T_N}\frac{1}{2}\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{knjn}\sigma_{kn}(s,T_N)\sigma_{jn}(s,T_N)ds\right)$$

$$\exp\left(\int_t^{T_M}\left(\frac{1}{2}\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{jnkn}\sigma_{jn}(s,T_M)\sigma_{kn}(s,T_M)-\sum_{k=1}^{K_r}\rho_{krX}\sigma_X(s)\sigma_{kr}(s,T_M)\right)ds\right)$$

$$\exp\left(\int_t^{T_M}\left(-\frac{1}{2}\sum_{k=1}^{K_r}\sum_{j=1}^{K_r}\rho_{krjr}\sigma_{jr}(s,T_M)\sigma_{kr}(s,T_M)-\frac{1}{2}\sigma_X^2(s)\right)ds\right)$$

$$E_t\left[\exp\left(\int_t^{T_M}\left(\sigma_X(s)dz_X(s)+\sum_{k=1}^{K_r}\sigma_{kr}(s,T_M)dz_{kr}(s)+\sum_{k=1}^{K_n}\{-\sigma_{kn}(s,T_M)\}dz_{kn}(s)\right)+\int_t^{T_N}\sum_{k=1}^{K_n}\sigma_{kn}(s,T_N)dz_{kn}(s)\right)\right]$$

Now the important thing is to calculate the expectation in equation (5),

$$E_t\left[\exp\left(\int_t^{T_M}\left(\sigma_X(s)dz_X(s)+\sum_{k=1}^{K_r}\sigma_{kr}(s,T_M)dz_{kr}(s)+\sum_{k=1}^{K_n}\{-\sigma_{kn}(s,T_M)\}dz_{kn}(s)\right)+\int_t^{T_N}\sum_{k=1}^{K_n}\sigma_{kn}(s,T_N)dz_{kn}(s)\right)\right]$$

$$=E_t\left[\exp\left(\int_t^{T_M}\left(\sum_{k=1}^{K_n}\{-\sigma_{kn}(s,T_M)\}dz_{kn}(s)+\sum_{k=1}^{K_n}\sigma_{kn}(s,T_N)dz_{kn}(s)+\sigma_X(s)dz_X(s)+\sum_{k=1}^{K_r}\sigma_{kr}(s,T_M)dz_{kr}(s)\right)\right)\right.$$

$$\left.E_{T_M}\left[\exp\left(\int_{T_M}^{T_N}\sum_{k=1}^{K_n}\sigma_{kn}(s,T_N)dz_{kn}(s)\right)\right]\right]$$

$$=\exp\left(\frac{1}{2}\int_{T_M}^{T_N}\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{knjn}\sigma_{kn}(s,T_N)\sigma_{jn}(s,T_N)ds\right) \tag{6}$$

$$E_t\left[\exp\left(\int_t^{T_M}\left(\sum_{k=1}^{K_n}\{-\sigma_{kn}(s,T_M)\}dz_{kn}(s)+\sum_{k=1}^{K_n}\sigma_{kn}(s,T_N)dz_{kn}(s)+\sigma_X(s)dz_X(s)+\sum_{k=1}^{K_r}\sigma_{kr}(s,T_M)dz_{kr}(s)\right)\right)\right]$$

Remark: Notice that we have used the tower property of expectations in the second line, and in the third line we have used a standard result for the expectation of the product of log-normally distributed random variables.

Now we can use the same standard result for the expectation in equation (6). Then we can combine equations (3), (5) and (6). After some algebra, we get

$$E_t\left[\exp\left(-\int_t^{T_N} r(s)\,ds\right)X(T_M)\right]=X(t)P_r(t,T_M)\frac{P(t,T_N)}{P(t,T_M)}\exp\left(\int_t^{T_M}C(s,T_M,T_N)ds\right)$$

where $\int_t^{T_M} C(s,T_M,T_N)ds$ is given by equation (4.22) and proposition 1 is proven.

Appendix 4 Proof of Proposition 2

Proof:

Our aim is to compute the expectation in equation (4.31) of proposition 2. The conceptual line of attack is, as in the proof of proposition 1, to reduce the problem to that of computing the expectation of the product of log-normally distributed random variables.

Using the tower property of expectations, we can write:

$$E_t\left[\exp\left(-\int_t^{T_{N_i}} r(s)ds\right)\frac{X(T_i)}{X(T_{i-1})}\right]=E_t\left[\exp\left(-\int_t^{T_{i-1}} r(s)ds\right)E_{T_{i-1}}\left[\exp\left(-\int_{T_{i-1}}^{T_{N_i}} r(s)ds\right)\frac{X(T_i)}{X(T_{i-1})}\right]\right]$$

But the following equation holds by proposition 1,

$$\begin{aligned}E_{T_{i-1}}\left[\exp\left(-\int_{T_{i-1}}^{T_{N_i}} r(s)ds\right)\left(\frac{X(T_i)}{X(T_{i-1})}\right)\right]&=\frac{1}{X(T_{i-1})}E_{T_{i-1}}\left[\exp\left(-\int_{T_{i-1}}^{T_{N_i}} r(s)ds\right)X(T_i)\right]\\&=P_r(T_{i-1},T_i)\frac{P\left(T_{i-1},T_{N_i}\right)}{P\left(T_{i-1},T_i\right)}\exp\left(\int_{T_{i-1}}^{T_i} C(s,T_i,T_{N_i})ds\right)\end{aligned}$$

Then the LHS of equation (4.31) becomes

$$\begin{aligned}&E_t\left[\exp\left(-\int_t^{T_{i-1}} r(s)ds\right)P_r(T_{i-1},T_i)\frac{P\left(T_{i-1},T_{N_i}\right)}{P\left(T_{i-1},T_i\right)}\exp\left(\int_{T_{i-1}}^{T_i} C(s,T_i,T_{N_i})ds\right)\right]\\&=\exp\left(\int_{T_{i-1}}^{T_i} C(s,T_i,T_{N_i})ds\right)E_t\left[\exp\left(-\int_t^{T_{i-1}} r(s)ds\right)P_r(T_{i-1},T_i)\frac{P\left(T_{i-1},T_{N_i}\right)}{P\left(T_{i-1},T_i\right)}\right]\end{aligned}$$

From equations (1), (3) and (4) of Appendix 1, we have:

$$E_t\left[\exp\left(-\int_t^{T_{i-1}} r(s)\,ds\right)P_r(T_{i-1},T_i)\frac{P\left(T_{i-1},T_{N_i}\right)}{P\left(T_{i-1},T_i\right)}\right]$$

$$=P(t,T_{i-1})\frac{P\left(t,T_{N_i}\right)}{P\left(t,T_i\right)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\exp\left(-\int_t^{T_{i-1}}\frac{1}{2}\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{knjn}\sigma_{kn}(s,T_{i-1})\sigma_{jn}(s,T_{i-1})ds\right)$$

$$\exp\left(\int_t^{T_{i-1}}\frac{1}{2}\sum_{k=1}^{K_n}\sum_{j=1}^{K_n}\rho_{knjn}\{\sigma_{kn}(s,T_i)\sigma_{jn}(s,T_i)-\sigma_{kn}\left(s,T_{N_i}\right)\sigma_{jn}\left(s,T_{N_i}\right)\}ds\right)$$

$$\exp\left(\int_t^{T_{i-1}}\sum_{k=1}^{K_r}\{\rho_{krX}\sigma_X(s)\sigma_{kr}(s,T_{i-1})-\rho_{krX}\sigma_X(s)\sigma_{kr}(s,T_i)\}ds\right)$$

$$\exp\left(\int_t^{T_{i-1}}\frac{1}{2}\sum_{k=1}^{K_r}\sum_{j=1}^{K_r}\rho_{krjr}\{\sigma_{kr}(s,T_{i-1})\sigma_{jr}(s,T_{i-1})-\sigma_{kr}(s,T_i)\sigma_{jr}(s,T_i)\}ds\right)$$

$$E_t\left[\exp\left(\left(\int_t^{T_{i-1}}\sum_{k=1}^{K_n}\sigma_{kn}(s,T_{i-1})dz_{kn}(s)+\sum_{j=1}^{K_r}\sigma_{rj}(s,T_i)dz_{rj}(s)\right)\right)\right.$$

$$\left.\exp\left(\int_t^{T_{i-1}}\left(\sum_{j=1}^{K_r}\{-\sigma_{rj}(s,T_{i-1})\}dz_{rj}(s)+\sum_{k=1}^{K_n}\sigma_{kn}\left(s,T_{N_i}\right)dz_{kn}(s)+\sum_{k=1}^{K_n}\{-\sigma_{kn}(s,T_i)\}dz_{kn}(s)\right)\right)\right]$$

Now we can, again, use a standard result for the expectation of the product of log-normally distributed random variables in order to compute the expectation in the last equation. After some tedious algebra, we obtain the RHS of equation (4.31). Hence proposition 2 is proven.

Appendix 5

Now we calculate the single term $E_{t_0}^{T^*}\left(\min\left(\max\left(\frac{X(T_i)}{X(T_{i-1})},1+F\right),1+C\right)\middle| w\right)$.

Step 1

Denote $X_i=\frac{X(T_i)}{X(T_{i-1})}$, then, firstly, we would like to compute the expectation of X_i. Girsanov's Theorem immediately shows that:

$$E_{t_0}^{T^*}\left[\frac{X(T_i)}{X(T_{i-1})}\right]=\frac{1}{P(t_0,T^*)}E_{t_0}\left[\exp\left(-\int_{t_0}^{T^*}r(s)ds\right)\frac{X(T_i)}{X(T_{i-1})}\right] \qquad (1)$$

We can now make use of our results in chapter 4.

When $i=1$, since $t_0 \equiv T_0$, the RHS of equation (1) is: $\dfrac{P_r(t_0,T_1)}{P(t_0,T_1)}\exp\left(\int_{t_0}^{T_1}C(s,T_1,T^*)ds\right)$.

When $i>1$, the RHS of equation (1) is:

$$\frac{P(t_0,T_{i-1})}{P(t_0,T_i)}\frac{P_r(t_0,T_i)}{P_r(t_0,T_{i-1})}\exp\left(\int_{T_{i-1}}^{T_i}C(s,T_i,T^*)ds+\int_{t_0}^{T_{i-1}}\{A(s,T_{i-1},T_i)+B(s,T_{i-1},T_i,T^*)\}ds\right).$$

Remark: The last but one formula follows from equation (4.21) and the last formula follows from equation (4.31).

Step 2

We can show that $X_i \equiv \dfrac{X(T_i)}{X(T_{i-1})}$ is log-normal. To see this, we recall equation (3.33)

and then change the probability measure to Q^{T^*}. We define

$$dz_{jn}^{T^*}(t)=dz_{jn}(t)-\sum_{k=1}^{K_n}\rho_{jnkn}\sigma_{kn}(t,T^*)dt, \qquad \text{for} \quad j=1,2,\dots K_n$$

$$dz_{jr}^{T^*}(t)=dz_{jr}(t)-\sum_{k=1}^{K_n}\rho_{knjr}\sigma_{kn}(t,T^*)dt, \qquad \text{for} \quad j=1,2,\dots K_r$$

$$dz_{X}^{T^*}(t)=dz_{X}(t)-\sum_{k=1}^{K_n}\rho_{knX}\sigma_{kn}(t,T^*)dt$$

where $z_{jn}^{T^*}(t)$, $z_{jr}^{T^*}(t)$ and $z_{X}^{T^*}(t)$ are Brownian motions in the measure Q^{T^*}. In this measure, $F_X(t,T_i)$ follows the stochastic equation:

$$\begin{aligned}\frac{dF_X(t,T_i)}{F_X(t,T_i)}&=\sigma_X(t)dz_X^{T^*}(t)+\sum_{k=1}^{K_r}\sigma_{kr}(t,T_i)dz_{kr}^{T^*}(t)-\sum_{k=1}^{K_n}\sigma_{kn}(t,T_i)dz_{kn}^{T^*}(t)\\&\quad+\operatorname{cov}\left(\frac{dP(t,T^*)}{P(t,T^*)}-\frac{dP(t,T_i)}{P(t,T_i)},\frac{dF(t,T_i)}{F(t,T_i)}\right)dt\\&\equiv\sigma_1(t)dW_1^{T^*}(t)+\operatorname{cov}\left(\frac{dP(t,T^*)}{P(t,T^*)}-\frac{dP(t,T_i)}{P(t,T_i)},\frac{dF(t,T_i)}{F(t,T_i)}\right)dt \qquad (2)\end{aligned}$$

where $W_1^{T^*}(t) = \left(\sigma_X(t) dz_X^{T^*}(t) + \sum_{k=1}^{K_r} \sigma_{kr}(t,T_i) dz_{kr}^{T^*}(t) - \sum_{k=1}^{K_n} \sigma_{kn}(t,T_i) dz_{kn}^{T^*}(t) \right) \Big/ \sigma_1(t)$ is a standard Brownian motion.

Solving equation (2), we get

$$F_X(T_i,T_i) = F_X(t,T_i) \exp\left(\int_t^{T_i} \sigma_1(s) dW_1^{T^*}(s) - \int_t^{T_i} \frac{1}{2} \sigma_1^2(s) ds \right)$$
$$\exp\left(\int_t^{T_i} \operatorname{cov}\left(\frac{dP(s,T^*)}{P(s,T^*)} - \frac{dP(s,T_i)}{P(s,T_i)}, \frac{dF(s,T_i)}{F(s,T_i)} \right) ds \right) \qquad (3)$$

Furthermore,

$$\begin{aligned} \frac{dF_X(t,T_{i-1})}{F_X(t,T_{i-1})} &= \sigma_X(t) dz_X^{T^*}(t) + \sum_{k=1}^{K_r} \sigma_{kr}(t,T_{i-1}) dz_{kr}^{T^*}(t) - \sum_{k=1}^{K_n} \sigma_{kn}(t,T_{i-1}) dz_{kn}^{T^*}(t) \\ &\quad + \operatorname{cov}\left(\frac{dP(t,T^*)}{P(t,T^*)} - \frac{dP(t,T_{i-1})}{P(t,T_{i-1})}, \frac{dF(t,T_{i-1})}{F(t,T_{i-1})} \right) dt \\ &\equiv \sigma_2(t) dW_2^{T^*}(t) + \operatorname{cov}\left(\frac{dP(t,T^*)}{P(t,T^*)} - \frac{dP(t,T_{i-1})}{P(t,T_{i-1})}, \frac{dF(t,T_{i-1})}{F(t,T_{i-1})} \right) dt \qquad (4) \end{aligned}$$

where $W_2^{T^*}(t) = \left(\sigma_X(t) dz_X^{T^*}(t) + \sum_{k=1}^{K_r} \sigma_{kr}(t,T_{i-1}) dz_{kr}^{T^*}(t) - \sum_{k=1}^{K_n} \sigma_{kn}(t,T_{i-1}) dz_{kn}^{T^*}(t) \right) \Big/ \sigma_2(t)$ is a standard Brownian motion.

Solving equation (4), we have

$$F_X(T_{i-1},T_{i-1}) = F_X(t,T_{i-1}) \exp\left(\int_t^{T_{i-1}} \sigma_2(s) dW_2^{T^*}(s) - \int_t^{T_{i-1}} \frac{1}{2} \sigma_2^2(s) ds \right)$$
$$\exp\left(\int_t^{T_{i-1}} \operatorname{cov}\left(\frac{dP(s,T^*)}{P(s,T^*)} - \frac{dP(s,T_{i-1})}{P(s,T_{i-1})}, \frac{dF(s,T_{i-1})}{F(s,T_{i-1})} \right) ds \right) \qquad (5)$$

Hence, we get the following expression for $\frac{X(T_i)}{X(T_{i-1})}$:

$$\begin{aligned}\frac{X(T_i)}{X(T_{i-1})} &= \frac{F_X\left(T_i,T_i\right)}{F_X\left(T_{i-1},T_{i-1}\right)} \\ &= \frac{P_r(t,T_i)}{P(t,T_i)}\frac{P(t,T_{i-1})}{P_r(t,T_{i-1})}\exp\left(\int_t^{T_i}\operatorname{cov}\left(\frac{dP(s,T^*)}{P(s,T^*)}-\frac{dP(s,T_i)}{P(s,T_i)},\frac{dF(s,T_i)}{F(s,T_i)}\right)ds\right) \\ &\quad \exp\left(-\int_t^{T_{i-1}}\operatorname{cov}\left(\frac{dP(s,T^*)}{P(s,T^*)}-\frac{dP(s,T_{i-1})}{P(s,T_{i-1})},\frac{dF(s,T_{i-1})}{F(s,T_{i-1})}\right)ds\right) \\ &\quad \exp\left(\int_t^{T_i}\sigma_1(s)dW_1^{T^*}(s)-\int_t^{T_i}\frac{1}{2}\sigma_1^2(s)ds\right)\exp\left(-\int_t^{T_{i-1}}\sigma_2(s)dW_2^{T^*}(s)+\int_t^{T_{i-1}}\frac{1}{2}\sigma_2^2(s)ds\right)\end{aligned}$$

This shows that $\ln X_i \equiv \ln\frac{X(T_i)}{X(T_{i-1})}$, for each i, is normally distributed.

Step 3 We wish to calculate the covariance matrix $\operatorname{cov}\left(\ln X_i,\ln X_j\right)$.

We can show that, when $j>i$,

$$\begin{aligned}&\operatorname{cov}\left(\ln X_i,\ln X_j\right) \\ &= \int_{t_0}^{T_{i-1}}\operatorname{cov}\left(\sum_{k=1}^{K_r}\left\{\sigma_{kr}(t,T_i)-\sigma_{kr}(t,T_{i-1})\right\}dz_{kr}^{T^*}-\sum_{p=1}^{K_n}\left\{\sigma_{pn}(t,T_i)-\sigma_{pn}(t,T_{i-1})\right\}dz_{pn}^{T^*},\right. \\ &\qquad\left.\sum_{k=1}^{K_r}\left\{\sigma_{kr}(t,T_j)-\sigma_{kr}(t,T_{j-1})\right\}dz_{kr}^{T^*}-\sum_{p=1}^{K_n}\left\{\sigma_{pn}(t,T_j)-\sigma_{pn}(t,T_{j-1})\right\}dz_{pn}^{T^*}\right)ds\,\cdot \\ &+\int_{T_{i-1}}^{T_i}\operatorname{cov}\left(\sigma_X dz_X^{T^*}+\sum_{k=1}^{K_r}\sigma_{kr}(s,T_i)dz_{kr}^{T^*}-\sum_{p=1}^{K_n}\sigma_{pn}(s,T_i)dz_{pn}^{T^*},\right. \\ &\qquad\left.\sum_{k=1}^{K_r}\left\{\sigma_{kr}(t,T_j)-\sigma_{kr}(t,T_{j-1})\right\}dz_{kr}^{T^*}-\sum_{p=1}^{K_n}\left\{\sigma_{pn}(t,T_j)-\sigma_{pn}(t,T_{j-1})\right\}dz_{pn}^{T^*}\right)ds\end{aligned}$$

Or when $j=i$,

$$\begin{aligned}\operatorname{var}\left[\ln X_i\right]\equiv\sigma_{\ln X_i}^2 &= \int_{t_0}^{T_{i-1}}\operatorname{var}\left(\sum_{k=1}^{K_r}\left\{\sigma_{kr}(t,T_i)-\sigma_{kr}(t,T_{i-1})\right\}dz_{kr}^{T^*}-\sum_{p=1}^{K_n}\left\{\sigma_{pn}(t,T_i)-\sigma_{pn}(t,T_{i-1})\right\}dz_{pn}^{T^*}\right)ds \\ &\quad+\int_{T_{i-1}}^{T_i}\operatorname{var}\left(\sigma_X dz_X^{T^*}+\sum_{k=1}^{K_r}\sigma_{kr}(s,T_i)dz_{kr}^{T^*}-\sum_{p=1}^{K_n}\sigma_{pn}(s,T_i)dz_{pn}^{T^*}\right)ds\end{aligned}$$

Step 4

We calculate the correlation between $\ln\hat{X}_i$ and the common factor w. Indeed, since $\ln \hat{X}_i \sim N(b_i, a_i^2)$, then $\mathrm{cov}(\ln \hat{X}_i, w) = \mathrm{cov}(a_i(\hat{a}_i\, w + \hat{d}_i\, \varepsilon_i), w) = a_i\, \hat{a}_i$ and hence the correlation between $\ln \hat{X}_i$ and w is $\hat{a}_i$.

From Ryten (2007), which, in turn, references Jackel (2004), we know that when $M > 2$, then we can approximate $\hat{a}_k$ by $\hat{a}_k \simeq \exp\left(\frac{1}{M-2}\left(\bar{k}_k - \frac{\sum_{i=1}^{M}\bar{k}_i}{2(M-1)}\right)\right)$, where $\bar{k}_k := \sum_{\substack{i\neq k\\ i=1}}^{M} \ln\left(\mathrm{cov}\left(\ln X_i, \ln X_k\right)\right)$, $k = 1,2,\ldots,M$.

For the cases when $M = 1$ or $M = 2$, we can show that: When $M = 1$, then $\hat{a}_1 = 1$; when $M = 2$, then $\hat{a}_1 = 1$, $\hat{a}_2 = \rho_{\ln X_1 \ln X_2}$, where $\rho_{\ln X_1 \ln X_2}$ is the correlation between $\ln X_1$ and $\ln X_2$. The case when $M = 2$ follows from Cholesky decomposition.

Appendix 6 Parameter Estimates

The parameters σ_{r1}, α_{r1}, σ_{n1}, α_{n1} are estimated using the Solver in Excel to run a cross sectional non linear regression based on equations (6.23) and (6.24) across the six different maturities. The parameters σ_X, ρ_{1rX}, ρ_{1nX}, ρ_{1r1n} are estimated from equations (6.25). All are estimated using monthly historical data.

One factor model

Table 2

correlation	Nominal 1	Real 1	CPI
Nominal 1	1	0.7504	0.018398
Real 1	0.7504	1	0.037818
CPI	0.018398	0.037818	1

Table 3

σ_{r1}	α_{r1}	σ_{n1}	α_{n1}	σ_X
0.006094	0.032193	0.007242	0.043585	0.0104

Two factor model

Table 4

correlation	Nominal 1	Nominal 2	Real 1	CPI
Nominal 1	1	-0.462963	0.5181	0.018398
Nominal 2	-0.462963	1	0.5181	0.018398
Real 1	0.5181	0.5181	1	0.037818
CPI	0.018398	0.018398	0.037818	1

Table 5

σ_{r1}	α_{r1}	σ_{n1}	α_{n1}	σ_{n2}	α_{n2}	σ_X
0.006094	0.032193	0.006498	0.064945	0.006332	0.000016	0.0104

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