Pricing exotic energy and commodity options in a multi-factor jump-diffusion model

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A multi-factor jump-diffusion model for commodities

This presentation draws on my papers "***A multi-factor jump-diffusion model for Commodities***" (submitted for publication), "***Commodity Options Optimised***" (Risk Magazine, May 2006, p72-77) and "***Pricing a class of exotic commodity options in a multi-factor jump-diffusion model***".

Fourier Transform methods

- Using Fourier Transform methods, can price (see Heston (1993), Duffie, Pan and Singleton (2000)):
- Standard European options
- Binary options

  - provided we know the characteristic function in analytical form.

- Our aim is to price a class of simple (European-style) exotic options which include options on:
- Difference (or ratio) of prices (either spot or futures prices) of two different commodities.
- Difference (or ratio) of prices of two futures contracts, either to different tenors or at different calendar times (eg cliquet type), on a single underlying commodity
- And some generalisations of the above.

Key Assumptions

- We assume the market is frictionless, (ie no bid-offer spreads, continuous trading is possible, etc) and arbitrage-free.
- No arbitrage => existence of an equivalent martingale measure (EMM).
- In this talk, we work exclusively under the EMM (or an EMM – it might not be unique).
- Note futures prices are martingales under the EMM. (Cox et al. (1981)).

Commodity prices

- Consider two (arbitrary) commodities, labelled 1 and 2 on which there are correspondingly two futures contracts. We denote the futures price of Commodity i $i=1,2$ at time t to time $T_{2,i}$ (ie the futures contract, into which Commodity i is deliverable, matures at time $T_{2,i}$) by $H_i(t,T_{2,i})$.

A class of exotic commodity options

- Our aim is to price a European-style option whose payoff is the greater of zero and a particular function involving the futures prices at times $T_{1,1}$ and $T_{1,2}$ of the futures contracts on Commodity 1 and Commodity 2 respectively.
- Choose $T_{1,2} \leq T_{1,1}$ (arbitrarily)
- The payoff is known at time $T_{1,1}$ but is paid at (a possibly later) time T_{pay} . Note $T_{pay} \geq T_{1,1} \geq T_{1,2}$.

A class of exotic commodity options

- More mathematically, we price a European-style option whose payoff is:

$$\max\left(\eta\left(\frac{H_1\left(T_{1,1}, T_{2,1}\right) - K^*\left[H_2\left(T_{1,2}, T_{2,2}\right)\right]^{\varepsilon}}{\left[H_2\left(T_{1,2}, T_{2,2}\right)\right]^{\alpha}} \right), 0 \right)$$

at time T_{pay}

where K^* is a constant which might, for example, account for different units of measurement.

A class of exotic commodity options

- Also $\eta = 1$ if the option is a call and $\eta = -1$ if the option is a put.

Note ε and α are constants (need not be integers).

Payoff again

- Payoff at time T_{pay}

$$\max\left(\eta\left(\frac{H_1\left(T_{1,1},T_{2,1}\right)-K^*\left[H_2\left(T_{1,2},T_{2,2}\right)\right]^{\varepsilon}}{\left[H_2\left(T_{1,2},T_{2,2}\right)\right]^{\alpha}}\right),0\right)$$

- Need $T_{pay} \geq T_{1,1} \geq T_{1,2}$, $T_{2,1} \geq T_{1,1}$, $T_{2,2} \geq T_{1,2}$
- Why consider this (slightly obscure) form?

Spread options

- General form: $\max\left(\eta\left(\frac{H_1(T_{1,1},T_{2,1})-K^*[H_2(T_{1,2},T_{2,2})]^{\varepsilon}}{[H_2(T_{1,2},T_{2,2})]^{\alpha}}\right),0\right)$

- Spread options on two different commodity futures: $\varepsilon=1$, $\alpha=0$
- Ratio spread or relative performance options on two different commodity futures: $\varepsilon=1$, $\alpha=1$
- For spread options on the spot, set

$$T_{1,1}\equiv T_{2,1}\equiv T_{1,2}\equiv T_{2,2}$$

Options on (slope of) futures curve

- General form: $$\max\left(\eta\left(\frac{H_1(T_{1,1},T_{2,1}) - K^*\left[H_2(T_{1,2},T_{2,2})\right]^{\varepsilon}}{\left[H_2(T_{1,2},T_{2,2})\right]^{\alpha}}\right),0\right)$$

- Could have $H_1(\bullet,\bullet) \equiv H_2(\bullet,\bullet)$ ie actual same underlying commodity.
- Spread options on futures commodity curve:

  $\varepsilon = 1$, $\alpha = 0$

- Ratio spread or relative performance options on futures commodity curve: $\varepsilon = 1$, $\alpha = 1$

Forward-start and cliquet options on futures

- General form: $\max\left(\eta\left(\frac{H_1(T_{1,1},T_{2,1})-K^*[H_2(T_{1,2},T_{2,2})]^{\varepsilon}}{[H_2(T_{1,2},T_{2,2})]^{\alpha}}\right),0\right)$
- Could have $H_1(\bullet,\bullet)\equiv H_2(\bullet,\bullet)$ ie actual same underlying commodity.
- Forward start options on futures prices:

  $T_{1,2}$ strictly $< T_{1,1}$, $\varepsilon=1$ $\alpha=0$
- Ratio forward start (ie single-leg cliquets) on futures prices:
- $T_{1,2}$ strictly $< T_{1,1}$, $\varepsilon=1$ $\alpha=1$

Forward-start and cliquet options on spot

- General form: $\max\left(\eta\left(\frac{H_1(T_{1,1},T_{2,1})-K^*[H_2(T_{1,2},T_{2,2})]^{\varepsilon}}{[H_2(T_{1,2},T_{2,2})]^{\alpha}}\right),0\right)$
- Again $H_1(\bullet,\bullet)\equiv H_2(\bullet,\bullet)$ ie actual same underlying commodity. Put $T_{1,2}\equiv T_{2,2}$ and $T_{1,1}\equiv T_{2,1}$
- Forward start options on spot: Again

  $T_{1,2}$ strictly $< T_{1,1}$, $\varepsilon=1$ $\alpha=0$
- Ratio forward start (ie single-leg cliquets) on spot: Again
- $T_{1,2}$ strictly $< T_{1,1}$, $\varepsilon=1$ $\alpha=1$

Return to general case

- General form: $\max\left(\eta\left(\frac{H_1(T_{1,1},T_{2,1})-K^*[H_2(T_{1,2},T_{2,2})]^{\varepsilon}}{[H_2(T_{1,2},T_{2,2})]^{\alpha}}\right),0\right)$

- How can we price these options (obviously we can do them using Monte Carlo but maybe there is another way)?

Stochastic Interest-rates

- We denote the (continuously compounded) risk-free short rate, at time t, by $r(t)$ and we denote the price of a zero coupon bond, at time t maturing at time T by $P(t,T)$.
- Interest-rates are assumed stochastic.

- Define for times $t_1 \geq t$ and $t_2 \geq t$

$$Y\left(t_1, T_{2,1}, t_2, T_{2,2}; t\right) \equiv \log\left(\frac{H_1\left(t_1, T_{2,1}\right)}{\left[H_2\left(t_2, T_{2,2}\right)\right]^{\varepsilon}} \Big/ \frac{H_1\left(t, T_{2,1}\right)}{\left[H_2\left(t, T_{2,2}\right)\right]^{\varepsilon}} \right)$$

The price of our option at time t is:

$$E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds \right) \max\left(\eta\left(\frac{H_1\left(T_{1,1}, T_{2,1}\right) - K^*\left[H_2\left(T_{1,2}, T_{2,2}\right)\right]^{\varepsilon}}{\left[H_2\left(T_{1,2}, T_{2,2}\right)\right]^{\alpha}} \right), 0 \right) \right]$$

- Which we can write as : $M_1 + M_2 - M_3$

where

$$M_1 \equiv \frac{(1+\eta)}{2} E_t \left[\exp\left(-\int_t^{T_{pay}} r(s)ds \right) H_1(T_{1,1}, T_{2,1}) [H_2(T_{1,2}, T_{2,2})]^{-\alpha} \right]$$

$$M_2 \equiv \frac{(1-\eta)}{2} E_t \left[\exp\left(-\int_t^{T_{pay}} r(s)ds \right) K^* [H_2(T_{1,2}, T_{2,2})]^{\varepsilon-\alpha} \right]$$

(work these out explicitly in the paper)

- And where

$$M_3 \equiv E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right)\left[H_2\left(T_{1,2},T_{2,2}\right)\right]^{\varepsilon-\alpha}\min\left(\right.\right.$$

$$\left.\left.\frac{H_1\left(t,T_{2,1}\right)}{\left[H_2\left(t,T_{2,2}\right)\right]^{\varepsilon}}\exp\left(Y\left(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t\right)\right),K^*\right)\right]$$

We can compute this with Fourier methods as follows:

Define

$$f\left(Y\left(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t\right)\right)\equiv \min\left(\frac{H_1\left(t,T_{2,1}\right)}{\left[H_2\left(t,T_{2,2}\right)\right]^{\varepsilon}}\exp\left(Y\left(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t\right)\right),K^*\right)$$

And then write it in terms of its F.T. $\hat{f}(z)$ ie

$$f\left(Y\left(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t\right)\right)=\frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty}\exp\left(-izY\left(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t\right)\right)\hat{f}(z)dz$$

Can show

$$\hat{f}(z)=\frac{H_1\left(t,T_{2,1}\right)}{\left[H_2\left(t,T_{2,2}\right)\right]^{\varepsilon}}\left(\frac{1}{z^2-iz}\right)\left(\frac{K^*\left[H_2\left(t,T_{2,2}\right)\right]^{\varepsilon}}{H_1\left(t,T_{2,1}\right)}\right)^{iz+1}$$

- Then

$$M_3 \equiv E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right)\left[H_2\left(T_{1,2},T_{2,2}\right)\right]^{\varepsilon-\alpha}\frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty}\exp\left(-izY\left(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t\right)\right)\hat{f}(z)dz\right]$$

$$=\frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty}E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right)\left[H_2\left(T_{1,2},T_{2,2}\right)\right]^{\varepsilon-\alpha}\exp\left(-izY\left(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t\right)\right)\right]\hat{f}(z)dz$$

$$\equiv\frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty}\Phi\left(-z;t,T_{1,1},T_{2,1},T_{1,2},T_{2,2}\right)\hat{f}(z)dz$$

- Where we call $\Phi\left(-z;t,T_{1,1},T_{2,1},T_{1,2},T_{2,2}\right)$ the "extended" characteristic function.

“Extended” characteristic function

- Explicitly:

$$\Phi\left(-z;t,T_{1,1},T_{2,1},T_{1,2},T_{2,2}\right)$$

$$=E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right)\left[H_2\left(T_{1,2},T_{2,2}\right)\right]^{\varepsilon-\alpha}\exp\left(-izY\left(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t\right)\right)\right]$$

Option price formula

- We have a formula for the option price, at time t :

$$M_1 + M_2 - \frac{1}{2\pi} \int_{iz_i - \infty}^{iz_i + \infty} \Phi\left(-z; t, T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}\right) \hat{f}(z) dz$$

- In fact, we can use symmetry to simplify the last term to an integral from 0 to infinity.
- We can always work out M_1 and M_2 explicitly if we can evaluate the "extended" characteristic function (by evaluating the "extended" characteristic function at z=i and z=0 respectively).

Option price formula (again)

- Again, the option price, at time t, is :

$$M_1 + M_2 - \frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} \Phi\left(-z; t, T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}\right)\hat{f}\left(z\right)dz$$

- This formula is valid for any underlying model for which we can evaluate the “extended” characteristic function.
- So this result should also be applicable for lots of different models. We’ll briefly look at one.

Stochastic Interest-rates

- We denote the (continuously compounded) risk-free short rate, at time t , by $r(t)$ and we denote the price of a zero coupon bond, at time t maturing at time T by $P(t,T)$. We assume that bond prices follow the extended Vasicek (Hull-White, 1-factor Gaussian HJM) process, namely,

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + \sigma_P(t,T)\,dz_P(t)$$

- $\sigma_P(t,T) \equiv \frac{\sigma_r}{\alpha_r}\left(1 - \exp\left(-\alpha_r(T-t)\right)\right)$ where

$\sigma_r > 0$ $\alpha_r > 0$ are constants.

(Actually can also do multi-factor Gaussian models)

Crosby (2005) model

- We work in the Crosby (2005) model which has the following main features:
- It is a no-arbitrage model for futures (or forward) commodity prices which automatically fits the initial term structure of futures (or forward) commodity prices.
- Consistent with mean reversion in (log) spot prices. We show in the papers that it is possible for jumps (depending upon their specification) to also contribute to mean reversion.

Crosby (2005) model

- Allows for jumps of two different types:

  The simpler type:

- Jumps which produce a parallel shift in the term structure of (log) futures commodity prices (more suitable for gold – "gold trades like a currency").

Crosby (2005) model

The more complicated type:

- Jumps which allow long-dated futures contracts to jump by less than short-dated futures contracts (especially suitable for natural gas, electricity and other energy-related commodities because this is line with empirical observations).
- This feature does not seem to have appeared in the literature before.
- Jumps of this type also contribute to mean reversion.
- Jumps of this type generate convenience yields which also have jumps (see papers for details).

- We label the two commodities, Commodity 1 and Commodity 2.
- Then for each i , we assume, as in the Crosby (2005) model, the dynamics of the futures prices of Commodity i under the EMM are:

$$\frac{dH_i(t,T)}{H_i(t-,T)} = \sum_{k=1}^{K_i} \sigma_{Hi,k}(t,T)dz_{Hi,k}(t) - \sigma_P(t,T)dz_P(t)$$

$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{i,mt}\exp\left(-\int_t^T b_{i,m}(u)du\right)\right)-1\right)dN_{mt} - \sum_{m=1}^{M} e_{i,m}(t,T)dt$$

- where

$$e_{i,m}(t,T) \equiv \lambda_m(t)E_{Nmt}\left(\exp\left(\gamma_{i,mt}\exp\left(-\int_t^T b_{i,m}(u)du\right)\right)-1\right)$$

- K_i is the number of Brownian factors (for example, 2 or 3).
- The form of the volatility functions $\sigma_{Hi,k}(t,T)$ can be somewhat general at this time but we assume they are deterministic (a specific form generates the effect of mean reversion, see paper).
- M is the number of Poisson processes.
- The Brownian motions can all be correlated (correlations are assumed constants).

Jump processes

- For each m , $m = 1,\ldots,M$, $\lambda_m(t)$ are the (assumed) deterministic intensity rates of the M Poisson processes.
- $b_{i,m}(u)$ for each m are non-negative deterministic functions. We call these the jump decay coefficient functions.
- $\gamma_{i,mt}$ are called the spot jump amplitudes.

Assumptions about the spot jump amplitudes $\gamma_{i,mt}$

- Assumption 2.1 in the papers:
- The spot jump amplitudes are (known) constants. Call these spot jump amplitudes $\beta_{i,m}$
- In this case, the jump decay coefficient functions $b_{i,m}(u)$ can be non-negative (but otherwise arbitrary) deterministic functions.
- (As an aside, we show in the papers that when $b_{i,m}(u) > 0$ then jumps also contribute to mean reversion and there is the effect that, after a jump, spot prices tend to revert back to a mean level.).

Assumptions about the spot jump amplitudes $\gamma_{i,mt}$

- Assumption 2.2 in the papers:
- In this case, the jump decay coefficient functions are set equal to zero. ie $b_{i,m}(t) \equiv 0$ for all t

In this case, the spot jump amplitudes can be random. Furthermore...

t

- For different m , the spot jump amplitudes are assumed to be independent of everything else. For each i , $i = 1,2$ but for a given m , we assume that, for this m , the spot jump amplitudes $\gamma_{i,mt}$ are normally distributed with mean $\beta_{i,m}$ and standard deviation $\upsilon_{i,m}$, and that the correlation between the spot jump amplitudes $\gamma_{1,mt}$ and $\gamma_{2,mt}$ is $\rho_{12,m}^{J}$.

- Note that the jump processes are modelled as common to both commodities but we can, of course, set a jump amplitude identically equal to zero to model the circumstance where one commodity jumps but the other one doesn’t.

Parallel shifts or exponentially dampened jumps

- Then by Ito: $d\left(\log H_i\left(t,T\right)\right)=$
  $-\frac{1}{2}(\text{instantaneous diffusion variance})\,dt$

$$+\sum_{k=1}^{K_i}\sigma_{Hi,k}\left(t,T\right)dz_{Hi,k}\left(t\right)-\sigma_P\left(t,T\right)dz_P\left(t\right)$$

$$+\sum_{m=1}^{M}\gamma_{i,mt}\exp\left(-\int_t^T b_{i,m}\left(u\right)du\right)dN_{mt}-\sum_{m=1}^{M}e_{i,m}\left(t,T\right)dt$$

“Primary” and “daughter” commodities

- It is common to talk about “primary” and “daughter” commodities.
- A “primary” commodity might be a very liquid and actively traded commodity such as WTI or Brent crude oil.
- A “daughter” commodity would then be a less actively traded blend of crude or a refined petroleum product such as heating oil, aviation fuel or gasoline.

Spread options

- This gives rise to crack spread options.
- We might also be interested in dark spread options (electricity minus coal) or spark spread options (electricity minus natural gas).
- Or perhaps options on seemingly unrelated commodities eg natural gas and a base metal.

Spread options

- Our model has lots of flexibility in choosing, eg diffusion volatilities $\sigma_{Hi,k}(t,T)$. We give a specific example in the paper of specifications which could be used for spread options.
- Can price these (and other options) using the Fourier technique described earlier.

- In the Crosby (2005) model, the “extended” characteristic function is conceptually straightforward (although rather long-winded). Unfortunately, it is also not analytic (if any of the jumps are of the type of assumption 2.1) as it involves a number of (at least) one dimensional integrals of the form...

$$\int_t^{T_{1,2}} \lambda_m(s)\exp\left(\left(\varepsilon-\alpha+iz\varepsilon\right)\beta_{2,m}\phi_{2,m}\left(s,T_{2,2}\right)-iz\beta_{1,m}\phi_{1,m}\left(s,T_{2,1}\right)\right)ds$$

where $\phi_{i,m}(s,T) \equiv \exp\left(-\int_s^T b_{i,m}(u)du\right)$

- Hence the option price involves at least a double (maybe even triple although, in practice, would choose $b_{i,m}(u)$ in a simple way eg constants) integral.

- Either: Bite the bullet and do the double (or possibly a triple) integral.
- Or: There is a nice simplifying assumption that one can make which makes evaluation into just a single one-dimensional (ie very fast) integral.

- Assume that, for each Poisson Process of type of assumption 2.1:
- The intensity rate is a constant
- And for both commodities, the jump decay coefficient functions are constants.
- And assume that for commodities which have common jumps, then they also have the same jump decay coefficient value (it can be shown that this would imply they are driven by common jump state variables and also common speed of “jump-reversion” after a jump).

- Then there is a power series expansion we can use which replaces the integral appearing in the "extended" characteristic function and is both much, much faster and much more accurate (eg 1 part in 10^{12} accuracy in very tiny fractions of a millisecond - the papers provide full details). Then option pricing is reduced to a single one-dimensional numerical integration
- This means however many Brownian motions and Poisson processes there are, option valuation is very fast.

Our F.T. technique is very general

- All you need is the ability to rapidly compute "extended" characteristic functions so our method has general interest (not just for commodities) eg Heston (1993), Levy processes like (stochastically time-changed) V.G. (Carr and co-authors (1999), (2005)).

Suggestions for further research

- This suggests avenues for further research:
- Replace Poisson jumps in case of assumption 2.2 (parallel shifts) by eg time-changed V.G. (seems straightforward – at least for a single commodity).
- Perhaps, replace Poisson jumps in case of assumption 2.1 (exponentially dampened) by eg time-changed V.G. (seems harder).

Summary

- The model is arbitrage-free.
- Automatically fits initial futures (or forward) commodity price curve.
- Captures empirical observations made about commodity prices (eg mean reversion, stochastic convenience yields (see paper)).
- Long-dated futures prices can jump less than short-dated futures. If so, jumps contribute to mean reversion.

Summary cont'd

- In the first paper, show we can price complex (exotic) commodity derivatives via Monte Carlo without discretisation error bias.
- Can rapidly price a class of simple European-style exotics including spread options, forward-start and cliquet options, in a consistent manner with a Fourier based algorithm.

Website for papers

- The papers which I mentioned earlier can be found on the website of the Centre for Financial Research at Cambridge University:

http://mahd-pc.jims.cam.ac.uk/seminar/2005.html

or, for the second paper, in Risk magazine, May 2006, "Commodity options optimised", p72-77.