A Multi-factor Jump-Diffusion Model for Commodities[1]

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Abstract

In this paper, we develop an arbitrage-free model for the pricing of commodity derivatives. The model generates futures (or forward) commodity prices consistent with any initial term structure. The model is consistent with mean reversion in commodity prices and also generates stochastic convenience yields. Our model is a multi-factor jump-diffusion model, one specification of which allows the prices of long-dated futures contracts to jump by smaller magnitudes than short-dated futures contracts, which, to our knowledge, is a feature that has not previously appeared in the literature, in spite of it being in line with stylised empirical observations (especially for energy-related commodities). Our model also allows for stochastic interest-rates. The model produces semi-analytic solutions for standard European options, which enable option prices to be evaluated in typically about $1/50^{th}$ of a second (depending upon parameter values and the required accuracy). This opens the possibility to calibrate the model parameters by deriving implied parameters from the market prices of options. We perform such a calibration on crude oil options and show that, allowing long-dated futures contracts to jump by smaller magnitudes than short-dated contracts, gives a greatly enhanced fit.

Keywords : Commodity options, commodity derivatives, jump-diffusion, mean reversion

1. Introduction

The aim of this paper is to develop an arbitrage-free multi-factor jump-diffusion model for pricing commodity options. The commodity could be, for example, crude oil, another petroleum product, gold, a base metal, natural gas or electricity.

Before turning our attention to commodities, it is worth reflecting on the development of interest-rate models. The paper by Vasicek (1977) introduced an equilibrium mean reverting interest-rate model into the literature. By introducing a time-dependent mean reversion level, this became the extended Vasicek model (Babbs (1990), Hull and White (1993)) which could automatically fit any initial term structure of interest-rates. These models focused principally on instantaneous short rates. Further research (Babbs (1990), Heath et al. (1992)) developed no-arbitrage models (including multi-factor versions) evolving the entire yield curve, consistent with its initial values. There are many parallels between the above interest-rate models and modelling futures (or forward) commodity prices. Some of the commodities literature (Gibson and Schwartz (1990)) has focused on equilibrium models with the first factor being the spot commodity price and the second factor being the instantaneous convenience yield whilst Schwartz (1997) introduced a third factor, with stochastic interest-rates. However these models leave the market price of convenience yield risk to be determined in equilibrium and are not necessarily consistent with any initial term structure of futures (or forward) prices. Subsequent models (by analogous techniques to interest-rate modelling) have been consistent with any initial term structure. See for example, Cortazar and Schwartz (1994), Carr and Jarrow (1995), Beaglehole and Chebanier (2002), Miltersen and Schwartz (1998), Miltersen (2003), Clewlow and Strickland (2000),(1999) with the latter particularly focussing on evolving the forward price curve. In this respect,

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[1] This is a condensed and enhanced version of a previous working paper (Crosby (2005)) entitled "Commodities: A simple multi-factor jump-diffusion model".

our paper is closest in spirit to Clewlow and Strickland (1999) though we also incorporate stochastic interest-rates and multiple jump processes with different specifications of jumps.

As a general rule, attention has mostly focused on pure diffusion models. Jumps were incorporated into interest-rate models in Babbs and Webber (1994),(1997), Bjork et al. (1997) and Jarrow and Madan (1995). See also Merton (1976),(1990), Hoogland et al. (2001), Duffie et al. (2000) and Runggaldier (2002). Paralleling these models, jumps have also been introduced into models for commodity prices in Hilliard and Reis (1998), Deng (1998), Clewlow and Strickland (2000), Benth et al. (2003) and Casassus and Collin-Dufresne (2005).

We will introduce a multi-factor jump-diffusion model which significantly extends existing models in the literature. Firstly, we outline some features of the commodities and commodity options markets.

It is an empirical fact (Bessembinder et al. (1995), Casassus and Collin-Dufresne (2005)) that the (spot) prices of most commodities seem to exhibit mean reversion[2].

Furthermore, it is also empirically observed that price jumps are usually both more frequent[3] and larger in magnitude in the commodities markets than in, for example, the equity or foreign exchange (fx) markets. Suppose there is a price jump in the spot fx rate, and we assume that there are no simultaneous jumps in the bond markets of the domestic and foreign currencies, then, clearly, the forward fx rate to all tenors must jump by the same proportional amount. When there is a jump in the price of gold, stylised observations suggest that the forward (or futures) gold price to all tenors do jump by the same proportional amount. In this respect, gold "trades like a currency". However, in most other commodities markets, especially in the case of crude oil, natural gas and electricity, stylised observations suggest a very different behaviour. The prices of short-dated (close to delivery) futures (or forward) contracts exhibit large jumps but the magnitude of these jumps is much lower for contracts with a greater time to delivery – indeed very long-dated contracts are often observed to scarcely jump at all, even when the very short-dated contracts have jumped by hundreds of per cent. Stylised observations also suggest that, after a large jump in the prices of these very short-dated contracts, prices often seem to revert to a more "usual" level rather quickly.

The empirically observed feature that the prices of short-dated futures (or forward) contracts jump by more than long-dated contracts is easily seen with two illustrations. The first illustration is from the historical prices of crude oil futures. The price of crude oil futures for next month delivery was approximately 17 dollars per barrel immediately before Iraq's invasion of Kuwait in August 1990 whilst the price of crude oil futures for 18 month delivery was approximately 19 dollars per barrel. Over the next 5 1/2 months (up to the start of the Gulf War), the (rolling) price for next month delivery, on two separate occasions, touched 40 dollars per barrel. In contrast, the (rolling) price of crude oil futures for 18 months delivery never went above 27 dollars per barrel during the whole period. Qualitatively similar behaviour has been observed (Geman (2005)) in the market for natural gas. Our second illustration, which is even more striking, is the case of electricity and can be found in Villaplana (2003). At the beginning of the second half of 1998, the prevailing price for electricity in the Pennsylvania-New Jersey-Maryland area of the U.S. was around 25 dollars per MWh for both spot (next day delivery) and one month forward delivery. During the course of the following six months, there were large jumps in the price of electricity which caused the spot price to rise above 350 dollars per MWh on three separate occasions. On each occasion that there was a jump in the spot price, a jump was also observed in the forward price of electricity. However, during this entire period, the forward price of electricity for one month delivery never exceeded 98 dollars per MWh. It is also striking to

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[2] Casassus and Collin-Dufresne (2005) show empirically that (spot) commodity prices exhibit a high degree of mean reversion in the real-world physical measure and also, albeit perhaps to a lesser degree, usually in the risk-neutral measure. They show how this possible difference in behaviour under the two different measures can be explained by state-dependent risk-premia. State-dependent risk-premia could also be used in precisely the same way in our model. However, since our aim is to develop a model for pricing commodity derivatives, we focus only on the risk-neutral measure and do not pursue the approach of Casassus and Collin-Dufresne (2005) here. Bessembinder et al. (1995) also find strong empirical evidence for mean reversion in the risk-neutral measure. In section 3, we will show that our model is consistent with mean reversion in the risk-neutral measure and that jumps may (depending upon their precise specification) also contribute to this effect.

[3] Of course, it is possible that the values of the intensity rates of the jumps and (if the jump amplitudes are random) the mean jump amplitudes are different (perhaps very different) in the real-world physical measure (in which the price jumps are observed) and in the risk-neutral measure (relevant for derivatives pricing), implying that jumps may be important in one measure but not the other. Casassus and Collin-Dufresne (2005) do not find any strong evidence for this and conclude (p2328) that "jumps would have a significant impact for the cross section of option prices". Of course, it would be grossly premature to conclude that the values under the two different measures would be identical or even approximately equal. In any event, since our aim is to price commodity derivatives, we will only be concerned with the values of these parameters in the risk-neutral measure.

observe that, on each of the three occasions the spot price jumped above 350 dollars per MWh, the spot price quickly (within two or three weeks) reverted back to a level below 40 dollars per MWh.

These illustrations demonstrate the importance of allowing prices of short-dated futures contracts (or forward prices) to jump by more then long-dated contracts within the context of historical price movements. One question that might be asked is: Is this important in the context of option pricing? We answer this question affirmatively in two ways. Firstly, in section 5, we will perform a calibration of our model to the market prices of options on crude oil futures which demonstrates that allowing for the effect of prices of short-dated futures contracts jumping by more than long-dated contracts gives a greatly improved fit. For the second way, suppose now we have our calibrated model parameters and we wish to price a European option, whose payoff is the greater of zero or the ratio of the price of a futures contract with a further (at option maturity) one month to delivery divided by the price of a futures contract with a further (at option maturity) two years to delivery minus a fixed strike. This is a simple type of exotic option on the slope of the term structure of futures commodity prices. If we assume (as the existing literature does) that, when there are jumps, futures contracts of all maturities jump by the same proportional amount then, it is easy to show that, the price of this exotic option, given the model parameters, is indifferent to jumps. However, if we assume that the prices of short-dated futures contracts jump by more than long-dated contracts, then the price of the option will be influenced by jumps, which is what one would expect to be the case given the empirically observed behaviour described above. This illustrates that the empirically observed historical price movements are important for pricing derivatives (we will outline in section 4 how our model can be used with Monte Carlo simulation to price exotic options – for a Fourier Transform based algorithm for pricing the type of exotic option just described, see Crosby (2006b)).

Despite the empirical evidence and its importance in pricing options, to our best knowledge, no previous papers have considered models for commodities which allow the prices of long-dated futures contracts to jump by smaller magnitudes than those of short-dated futures contracts.

We also observe that the market prices of options on many commodities imply Black (1976) volatilities which vary with the strike of the option ie market prices imply a volatility skew (or smile). Furthermore, the volatility skew is much, much more pronounced for options with shorter maturities (Geman (2005) and Geman and Nguyen (2003) provide empirical data which demonstrates how marked this behaviour is, particularly for energy-related commodities) and it is well known that such behaviour could be accounted for through a jump-diffusion model. Additionally, we note (see Geman (2005)) that the implied volatilities of commodity options typically decrease with increasing option maturity (this is very marked for energy-related commodities).

We would like our model to incorporate all of the above stylised observations of the commodities and commodity options markets. Examination of our model will show that indeed it does.

We will deliberately develop our model with considerable generality so that it is flexible enough to be applied to options on almost any commodity. For example, the implied volatilities of options on some commodities (for example, natural gas) exhibit seasonality[4]. The volatility specification we will use is flexible enough to model this when appropriate (for example, by utilising the specific form in Miltersen (2003)). Similarly, our model will be general enough to cater for jumps more appropriate for gold as well as for jumps more appropriate for crude oil, natural gas and electricity (as outlined earlier).

We will also briefly discuss convenience yields in our model. In the literature, convenience yields are often related to cost of storage. We will not explicitly do this because it does not appear helpful for modelling, for example, the price of an option on electricity which is extremely expensive to store. Instead, we will postulate the dynamics of futures commodity prices and then infer from them the dynamics of convenience yields. We will show that, in our model, convenience yields (in general) can exhibit jumps and furthermore that jumps in convenience yields are intrinsically linked with the ability to capture the effect of short-dated futures contracts jumping by more than long-dated futures contracts.

We will assume that interest-rates are stochastic. When interest-rates are stochastic, futures commodity prices and forward commodity prices are no longer the same. In this paper, we will work with both futures and forward prices but mostly with futures commodity prices.

We will assume that markets are frictionless. That is, continuous trading is possible and we assume that there are no bid-offer spreads in the commodities markets or in the bond markets. Of course, we do not assume that the commodity can be stored or insured without cost since it is precisely these costs which give rise to the notion of convenience yield.

We will assume that markets are free of arbitrage.

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[4] Note that our model is automatically consistent with any seasonality in the term structure of futures (or forward) commodity prices since it is consistent with any given initial term structure.

It is well known (Harrison and Pliska (1981), Duffie (1996)) that, under these assumptions, there exists an equivalent martingale measure under which futures prices are martingales. In the case of a diffusion model, if there are sufficient futures (or forward) contracts (and risk-free bonds) traded, then any derivative can be instantaneously hedged or replicated by a dynamic self-financing portfolio of futures contracts (and risk-free bonds). The market would thus be complete. In this case, the equivalent martingale measure is unique. However, in the case of a jump-diffusion model, the market may be either complete or incomplete. If the market is incomplete then the equivalent martingale measure would not be unique. In the case of incompleteness, we will assume that an equivalent martingale measure is "fixed by the market" through the market prices of options and we will call this (by an abuse of language but for the sake of brevity) the (rather than an) equivalent martingale measure. It is also possible for our jump-diffusion model to lead to a market which is complete. The circumstances in which our jump-diffusion model gives rise to a complete market are specified in section 2.

The remainder of this paper is structured as follows. In section 2, we will provide notation and introduce the model. In section 3, we will relate it to stochastic convenience yields and to mean reversion. In section 4, we will show how the model can be used in connection with Monte Carlo simulation to price complex (exotic) commodity derivatives. In section 5, we will derive the prices of standard options, in semi-analytical form, illustrate our model with five examples and calibrate it to the market prices of options on crude oil futures. Section 6 is a short conclusion.

2. The model of futures commodity prices

Notation:

Let us explain some notation. All jump-diffusion processes are assumed right continuous. The value of $H(t,T)$ just before a jump at time t is $H(t-,T) = \lim_{u \uparrow t} H(u,T)$. When, for the sake of brevity, we write $\dfrac{dH(t,T)}{H(t,T)}$ in a SDE, we strictly mean $\dfrac{dH(t,T)}{H(t-,T)}$.

We define today to be time t_0 and we denote calendar time by t, $(t \geq t_0)$.

In this and subsequent sections, we will work exclusively in the equivalent martingale measure which, as already indicated, may, in fact, not be unique. If it is not unique, we will assume that one has been "fixed by the market" and we will call this the (rather than an) equivalent martingale measure. We denote the equivalent martingale measure by Q. We fix a probability space $(\Omega, \Im, Q)$ and an information filtration $(\Im_t)_{t \geq t_0}$ which we assume satisfies the usual conditions. We denote expectations, at time t, with respect to the equivalent martingale measure Q by $E_t[\ ]$.

Stochastic evolution of interest-rates:

We assume that interest-rates in our model are stochastic. Let us introduce some notation. We denote the (continuously compounded) risk-free short rate, at time t, by $r(t)$ and we denote the price, at time t, of a (credit risk free) zero coupon bond maturing at time T by $P(t,T)$.

We assume that (under the equivalent martingale measure Q) the short rate follows the extended Vasicek (one factor Gaussian) process, (Babbs (1990), Hull and White (1990),(1993)) namely,

$dr(t) = \alpha_r(\gamma(t) - r(t))dt - \sigma_r dz_P(t)$, where $dz_P(t)$ denotes standard Brownian increments,

or equivalently (Babbs (1990), Heath et al. (1992)) the dynamics of bond prices are

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \sigma_P(t,T)dz_P(t), \text{ where } \sigma_P(t,T) \equiv \frac{\sigma_r}{\alpha_r}\left(1 - \exp\left(-\alpha_r(T-t)\right)\right), \quad (2.1)$$

where σ_r and α_r are positive, finite constants and $\gamma(t)$ is defined so as to be consistent with the initial term structure (ie the term structure of interest rates today, time t_0), which we take as given.

Commodities:

We denote the value of the commodity, at time t, by C_t. The value of the commodity is usually termed the spot price. However, in this paper, we shall often use the expression "value of the commodity" because, in some commodities markets, the spot price is not always exactly easy to define. We denote the forward commodity price, at time t, to (ie for delivery at) time T, by $F(t,T)$. We denote the futures commodity price, at time t, to (ie the futures contract matures at) time T, by $H(t,T)$.

It can be shown (Cox et al. (1981), Duffie (1996)), that in the absence of arbitrage, that

$$F(t,T) = \frac{E_t\left[\exp\left(-\int_t^T r(s)\,ds\right)C_T\right]}{P(t,T)} \tag{2.2}$$

and $H(t,T) = E_t[C_T]$. $\tag{2.3}$

A key to modelling commodity prices when interest-rates are stochastic is to recognise that, in this case, futures commodity prices and forward commodity prices are not the same. Indeed equations 2.2 and 2.3 show that futures prices are martingales with respect to the equivalent martingale measure whereas, when interest-rates are stochastic, forward prices are not.

Note that equations 2.2 and 2.3 are consistent with

$$F(t,t) = C_t = H(t,t) \text{ and } F(T,T) = C_T = H(T,T) \tag{2.4}$$

We take as given our initial term structure (ie the term structure today, time t_0) of futures commodity prices. That is, we know $H(t_0,T)$ for all T of interest, ($T \geq t_0$) (perhaps, in practice, by interpolation of the futures prices of a finite number of futures contracts).

In some models, the dynamics of the value of the commodity are posited and then equations 2.2 and 2.3 would be used to derive the dynamics of forward commodity prices and futures commodity prices. By contrast, our model will posit the dynamics of futures commodity prices. In other words, futures contracts are not derivatives but, instead, are the primitive assets of our model.

The dynamics of futures commodity prices $H(t,T)$, under the equivalent martingale measure, which we will posit, will be consistent with the martingale property of equation 2.3.

Now we introduce the instantaneous futures convenience yield forward rate $\varepsilon(t,T)$, at time t, to time T via the relation $H(t,T) = \frac{C_t}{P(t,T)}\exp\left(-\int_{s=t}^T \varepsilon(t,s)\,ds\right)$. $\tag{2.5}$

We introduce K standard Brownian increments denoted by $dz_{Hk}(t)$, for each k, $k = 1,2,...,K$. We denote the correlation between $dz_P(t)$ and $dz_{Hk}(t)$ by ρ_{PHk}, for each k, and the correlation between $dz_{Hk}(t)$ and $dz_{Hj}(t)$ by ρ_{HkHj} for each j and k, $j,k = 1,2,...,K$.

We also introduce M independent Poisson processes denoted by N_{mt}, for each m, $m = 1,...,M$, with $N_{mt_0} \equiv 0$, whose intensity rates are $\lambda_m(t)$ under the equivalent martingale

measure Q. We assume that, for each m, $m=1,...,M$, $\lambda_m(t)$ are positive, bounded deterministic functions of at most t.

We introduce $b_m(t)$, for each m, $m=1,...,M$, which are non-negative deterministic functions, which we call jump decay coefficient functions. They control, when there are jumps, how much less long-dated futures contracts jump by than short-dated futures contracts, in a way which we make precise in assumption 2.5 and remark 2.11. We introduce γ_{mt}, for each m, $m=1,...,M$, which we call spot jump amplitudes (in view of remark 2.11), which are parameters, which determine the size of the jump, conditional on a jump in N_{mt}. At risk of complication, but for the sake of brevity, we will consider two possible specifications for the spot jump amplitudes, and in turn, these are linked to two possible specifications of the jump decay coefficient functions.

For each m, $m=1,...,M$, we assume that either:

Assumption 2.1 :

The spot jump amplitudes γ_{mt} are assumed to be finite constants. In this case, the jump decay coefficient functions $b_m(t)$ are assumed to be any non-negative deterministic functions. $\qquad\bullet$

Or:

Assumption 2.2 :

The spot jump amplitudes γ_{mt} are assumed to be independent and identically distributed random variables, whose distribution is defined with respect to the equivalent martingale measure Q, satisfying $0 < E_{Nmt}\left(\exp(\gamma_{mt})\right) < \infty$, each of which is independent of each of the Brownian motions and of each of the Poisson processes. In this case, the jump decay coefficient functions $b_m(t)$ are assumed to be identically equal to zero ie $b_m(t) \equiv 0$ for all t. $\qquad\bullet$

where, for each m, E_{Nmt} denotes the expectation operator, at time t, conditional on a jump occurring in N_{mt}. If, for a given m, the spot jump amplitude is constant (assumption 2.1), the expectation operator is set equal to its argument (see equation 2.6).

Remark 2.3 : Note that for each m, we assume either assumption 2.1 or assumption 2.2 is satisfied. For different m it could be a different assumption (ie if $M>1$, we can mix the assumptions). Note also that although we index the spot jump amplitudes γ_{mt} with t, both assumptions imply that their outcomes do not depend on t, ie the index simply refers to the time at which a jump may occur.

Remark 2.4 : The motivation for these assumptions is as follows: Crosby (2005) and (in the context of an interest-rate model) Bjork et al. (1997) show that it may not be possible, in general, unless a very specific and non-trivial condition is satisfied, whilst being consistent with the absence of arbitrage, to have both jumps whose amplitudes are random variables and simultaneously have jump decay coefficient functions ($b_m(t)$) which are not identically zero. Hence we assume that all the Poisson processes satisfy either assumption 2.1 or assumption 2.2. We will develop the model with assumptions 2.1 and 2.2 in parallel since the choice of these assumptions scarcely alters the development.

We are motivated by the presence of $P(t,T)$ in the denominator of equation 2.5 (see also comments in section 3), the effect of applying Ito's lemma to $\dfrac{C_t}{P(t,T)}\exp\left(-\displaystyle\int_{s=t}^{T}\varepsilon(t,s)\,ds\right)$ and by the knowledge that futures commodity prices are martingales in the equivalent martingale measure.

Assumption 2.5 :

We assume that the dynamics of futures prices in the equivalent martingale measure Q are:

$$\frac{dH(t,T)}{H(t,T)} = \sum_{k=1}^{K} \sigma_{Hk}(t,T) dz_{Hk}(t) - \sigma_P(t,T) dz_P(t)$$

$$+ \sum_{m=1}^{M} \left(\exp\left(\gamma_{mt} \exp\left(-\int_t^T b_m(u)\, du \right) \right) - 1 \right) dN_{mt}$$

$$- \sum_{m=1}^{M} \lambda_m(t) E_{Nmt} \left(\exp\left(\gamma_{mt} \exp\left(-\int_t^T b_m(u)\, du \right) \right) - 1 \right) dt \qquad (2.6)$$

where $\sigma_{Hk}(t,T)$, for each k, $k = 1,2,...,K$, are bounded (ie satisfying the Novikov condition) deterministic functions of t and T, of the form:

$$\sigma_{Hk}(t,T) = \eta_{Hk}(t) + \chi_{Hk}(t) \exp\left(-\int_t^T a_{Hk}(u)\, du \right), \qquad (2.7)$$

where $\eta_{Hk}(t)$, $\chi_{Hk}(t)$ and $a_{Hk}(u)$ are deterministic functions[5]. •

Remark 2.6 : Futures commodity prices are martingales in the equivalent martingale measure Q.

Remark 2.7 : In the absence of jumps, the dynamics of futures commodity prices in the equivalent martingale measure are very similar to those of forward prices in Clewlow and Strickland (1999) (although we also incorporate stochastic interest-rates). When $K = 2$ (and in the absence of jumps), equation 2.6 gives dynamics for futures commodity prices which are essentially identical to those in Miltersen and Schwartz (1998) although they make the starting point of their model, the dynamics of spot commodity prices and convenience yields. Thus, equation 2.6 generalises well known diffusion models in the literature. It also generalises Casassus and Collin-Dufresne (2005), Clewlow and Strickland (2000) and Hilliard and Reis (1998) in two main ways. Firstly, our model is automatically consistent with any initial term structure of futures commodity prices. Secondly, our model has a much more general specification of jumps. For example, the former models only consider jumps of the type in assumption 2.2. We explain in remark 3.3 why these latter models cannot generate jumps of the type in assumption 2.1.

Remark 2.8 : If the spot jump amplitudes for all m, $m = 1,...,M$, are constants (ie they all satisfy assumption 2.1), and if the the number of futures contracts in the market is greater than or equal to $K + M + 1$, then the results of Babbs and Webber (1994),(1997), Bjork et al. (1997), Jarrow and Madan (1995), Hoogland et al. (2001) and Crosby (2005) show that our market is complete. In this case, there exists a unique equivalent martingale measure. All derivatives can be replicated or hedged (see, for example, Hoogland et al. (2001) and Crosby (2005) for more details) by a self-financing portfolio of futures contracts and risk-free bonds. This will imply the unique pricing of all derivatives.

Remark 2.9 : In practise, most commodities markets have futures contracts of many different maturities. For example, there are futures contracts on WTI grade crude oil for more than 120 different maturities. Even for base metals, which are less actively traded than crude oil, the London Metal Exchange trades futures contracts for 27 different maturities on a wide variety of different base metals. Hence, in the case that all spot jump amplitudes are constants (ie the spot jump amplitudes for all m satisfy assumption 2.1), then, for example, if K were set equal to three, then the number of Poisson processes M could be set to 20 or more and, in this case, our market would still be complete. In

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[5] We note that it will become clear later that in order to avoid a potential degeneracy we may put $\eta_{Hk}(t) \equiv 0$ for all k except one, (or combine terms of the form $\eta_{Hk}(t) dz_{Hk}(t)$) but we will write out equations below in full to ease notation. In addition, we note that the functions $a_{Hk}(u)$ will be seen to have an obvious interpretation as mean reversion rate parameters and hence we expect them to be positive.

practise, setting M to equal just one or two, say, would be more realistic to allow for an easier calibration whilst still allowing considerable flexibility in the model.

Remark 2.10 : If any of the spot jump amplitudes, for any m, $m = 1, ..., M$, satisfy assumption 2.2 (ie if any are random variables), or if there are insufficient futures contracts in the market, then the same references, cited in remark 2.8, imply that our market will not be complete. The absence of arbitrage implies (Harrison and Pliska (1981)) that an equivalent martingale measure exists but the incompleteness of our market implies it will not be unique and hence we cannot uniquely determine the price of a derivative since embedded within the values of the intensity rates $\lambda_m(t)$ and (in the case of assumption 2.2) the parameters of the distribution of the spot jump amplitudes, under an equivalent martingale measure, are market prices of risk. Corresponding to each possible equivalent martingale measure, there will be different values of these parameters leading to different derivative prices. In section 5, we show that standard options have prices of a simple form. We can estimate the parameters of our model, by inverting the market prices of such options (provided there are sufficient options in the market). Embedded within those parameters, specifically the intensity rates and (for assumption 2.2) the parameters of the distribution of the spot jump amplitudes, are market prices of risk which are "fixed by the market" and which therefore also "fix" the equivalent martingale measure Q. This is a standard technique in incomplete markets.

For notational convenience, we define, for each m, the deterministic function $e_m(t,T)$ via:

$$e_m(t,T) \equiv \lambda_m(t) E_{Nmt}\left(\exp\left(\gamma_{mt} \exp\left(-\int_t^T b_m(u)\,du \right) \right) - 1 \right) \tag{2.8}$$

By Ito's lemma for jump-diffusions, applied to equation 2.6, and using equation 2.8,

$$d\left(\ln H(t,T)\right) = -\frac{1}{2}\left\{ \sum_{k=1}^K \sigma_{Hk}^2(t,T) + \sigma_P^2(t,T) - 2\sum_{k=1}^K \rho_{PHk}\sigma_P(t,T)\sigma_{Hk}(t,T) \right\}dt$$

$$-\frac{1}{2}\left\{ \sum_{k=1}^K \sum_{j=1}^{k-1} 2\rho_{HkHj}\sigma_{Hk}(t,T)\sigma_{Hj}(t,T) \right\}dt + \sum_{k=1}^K \sigma_{Hk}(t,T)dz_{Hk}(t) - \sigma_P(t,T)dz_P(t)$$

$$+\sum_{m=1}^M \gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du \right)dN_{mt} - \sum_{m=1}^M e_m(t,T)\,dt \tag{2.9}$$

and where we have used the usual convention that if the upper index is strictly less than the lower index in a summation, then the sum is set to zero.

Remark 2.11 : Equation 2.9 enables us to better describe the size of the jump when one happens. When there is a jump in N_{mt}, $\ln H(t,T)$ changes by $\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du \right)$. Let us briefly consider the implications of this. When there is a jump, the log of the futures commodity prices infinitesimally close to maturity (ie the spot price $C_t \equiv \ln H(t,t)$) jump by γ_{mt}. However, the log of the futures commodity prices for delivery $(T-t)$ years ahead jump by $\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du \right)$.

Considering the limit, as $(T-t) \to \infty$, (and provided $\exp\left(-\int_t^T b_m(u)\,du \right) \to 0$), then very long-dated futures commodity prices do not jump at all. The effect of the jump decay coefficient function $b_m(t)$, (which is assumed always non-negative), is to exponentially dampen the effect of the jump through futures commodity price tenor. This seems to be in line with empirical observations in most

commodities markets (this is particularly a feature in the case of oil, natural gas and electricity). In the case of assumption 2.2, $b_m(t) \equiv 0$ and jumps cause parallel shifts in the log of the futures commodity prices across different tenors (stylised observations suggest this is more appropriate for gold).

Let us return to the model: Define the state variables:

$$X_P(t) \equiv \int_{t_0}^{t} \sigma_r \exp\left(-\alpha_r(t-s)\right) dz_P(s), \; Y_P(t) \equiv \int_{t_0}^{t} \sigma_r dz_P(s), \; \text{and, for each } k, \; k=1,2,...,K,$$

$$X_{Hk}(t) \equiv \int_{t_0}^{t} \chi_{Hk}(s) \exp\left(-\int_s^t a_{Hk}(u) du\right) dz_{Hk}(s), \; Y_{Hk}(t) \equiv \int_{t_0}^{t} \eta_{Hk}(s) dz_{Hk}(s). \quad (2.10)$$

Define, for each m, $m=1,...,M$, the jump state variables:

$$X_{Nm}(t) \equiv \int_{t_0}^{t} \gamma_{ms} \exp\left(-\int_s^t b_m(u) du\right) dN_{ms}. \quad (2.11)$$

Proposition 2.12 : The evolution, from time t_0 to time t, of the futures commodity price to time T, can be expressed in terms of the state variables as:

$$H(t,T) = H(t_0,T) \exp\left(\int_{t_0}^{t} -\frac{1}{2}\left\{\sum_{k=1}^{K} \sigma_{Hk}^2(s,T) + \sigma_P^2(s,T) - 2\sum_{k=1}^{K} \rho_{PHk} \sigma_P(s,T) \sigma_{Hk}(s,T)\right\} ds\right)$$

$$\exp\left(\int_{t_0}^{t} -\frac{1}{2}\left\{\sum_{k=1}^{K}\sum_{j=1}^{k-1} 2\rho_{HkHj} \sigma_{Hk}(s,T) \sigma_{Hj}(s,T)\right\} ds\right)$$

$$\exp\left(\sum_{k=1}^{K} Y_{Hk}(t) + \sum_{k=1}^{K}\left[\exp\left(-\int_t^T a_{Hk}(u) du\right) X_{Hk}(t)\right] + \frac{\exp\left(-\alpha_r(T-t)\right)}{\alpha_r} X_P(t) - \frac{1}{\alpha_r} Y_P(t)\right)$$

$$\exp\left(\sum_{m=1}^{M}\left(\exp\left(-\int_t^T b_m(u) du\right) X_{Nm}(t)\right) - \sum_{m=1}^{M}\int_{t_0}^{t} e_m(s,T) ds\right) \quad (2.12)$$

Proof : Rewrite equation 2.9 in integral form from t_0 to t, then use equations 2.10 and 2.11. •

This shows that $H(t,T)$ and $H(t,t) \equiv C_t$ are Markov in a finite number of state variables[6].

Remark 2.13 : With the help of results in section 4 (specifically equation 4.5), it is straightforward to verify by direct calculation, using equation 2.12, that

$E_t[C_T] \equiv E_t[H(T,T)] = H(t,T)$ which confirms consistency with equation 2.3.

Proposition 2.14 : The forward commodity price $F(t,T)$, at time t, to (ie for delivery at) time T is related to the futures commodity price $H(t,T)$ via:

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[6]We note that it is straightforward to combine the $Y_{Hk}(t)$ and $Y_P(t)$ into a single state variable. We could do this, but prefer not to, in order to maximise the intuition behind the model. However, it shows that $H(t,t) \equiv C_t$ and $H(t,T)$ are, in fact, Markovian in $K+2+M$ state variables.

$$F(t,T) = H(t,T)\exp\left(\int_t^T \left(\sum_{k=1}^K \rho_{PHk}\sigma_P(s,T)\sigma_{Hk}(s,T) - \sigma_P^2(s,T)\right)ds\right). \tag{2.13}$$

Proof : We can change the probability measure in equation 2.2 to the forward risk-adjusted measure under which forward commodity prices are martingales and then use Girsanov's theorem. Since Jamshidian (1993) (in a diffusion setting) proves a similar result, we omit the full proof here. ●

Remark 2.15 : Note that forward commodity prices and futures commodity prices differ only by a deterministic term. Our model has, thus far, been expressed in terms of futures commodity prices, but, clearly, it would have been straightforward to have worked with forward commodity prices instead.

3. Stochastic convenience yields and mean reverting commodity prices

Our aim in this section is to give greater insight into our model and, in particular, to give results about stochastic convenience yields and mean reversion[7] which show that our model is able to capture the stylised observations of the commodities markets that were made in section 1.

Firstly, we derive the dynamics of the value of the commodity.

Proposition 3.1 : The dynamics of the value of the commodity are as follows. If we define

$$\varepsilon_r(t) \equiv \varepsilon(t_0,t) - \left(\int_{t_0}^t -\frac{1}{2}\left\{\sum_{k=1}^K \left(2\sigma_{Hk}(s,t)\frac{\partial\sigma_{Hk}(s,t)}{\partial t}\right) + 4\sigma_P(s,t)\frac{\partial\sigma_P(s,t)}{\partial t}\right\}ds\right)$$

$$-\left(\int_{t_0}^t -\frac{1}{2}\left\{\sum_{k=1}^K\sum_{j=1}^{k-1}2\rho_{HkHj}\left(\sigma_{Hk}(s,t)\frac{\partial\sigma_{Hj}(s,t)}{\partial t} + \sigma_{Hj}(s,t)\frac{\partial\sigma_{Hk}(s,t)}{\partial t}\right)\right\}ds\right)$$

$$-\left(\int_{t_0}^t \frac{1}{2}\left\{\sum_{k=1}^K 2\rho_{PHk}\sigma_P(s,t)\frac{\partial\sigma_{Hk}(s,t)}{\partial t}\right\}ds\right) - \left(\int_{t_0}^t \frac{1}{2}\left\{\sum_{k=1}^K 2\rho_{PHk}\sigma_{Hk}(s,t)\frac{\partial\sigma_P(s,t)}{\partial t}\right\}ds\right)$$

$$-\left(\int_{t_0}^t \sum_{k=1}^K \frac{\partial\sigma_{Hk}(s,t)}{\partial t}dz_{Hk}(s)\right)$$

$$+\left(\int_{t_0}^t \sum_{m=1}^M \gamma_{ms}b_m(t)\exp\left(-\int_s^t b_m(u)du\right)dN_{ms}\right) + \left(\int_{t_0}^t \sum_{m=1}^M \frac{\partial e_m(s,t)}{\partial t}ds\right) \tag{3.1}$$

then

$$\frac{dC_t}{C_t} = \left(r(t) - \varepsilon_r(t)\right)dt + \sum_{k=1}^K \sigma_{Hk}(t,t)dz_{Hk}(t)$$

$$+\sum_{m=1}^M \left(\exp(\gamma_{mt}) - 1\right)dN_{mt} - \left(\sum_{m=1}^M e_m(t,t)\right)dt. \tag{3.2}$$

Proof : We rewrite equation 2.9, our SDE for $\ln H(t,T)$, for $\ln H(s,t)$ instead, and then rewrite in integral form from t_0 to t. Then by differentiating with respect to t, we get the SDE for the dynamics

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[7] Casassus and Collin-Dufresne (2005) show, by using state-dependent risk-premia, that there can be a different degree of mean reverting behaviour between the real-world physical measure and the risk-neutral measure. State-dependent risk-premia could also be used in precisely the same way in our model. However, for the sake of brevity and since our present aim is to develop a model for pricing commodity derivatives, we focus only on the risk-neutral equivalent martingale measure. We will show that our model produces mean reversion in the risk-neutral equivalent martingale measure (except in the (degenerate) case that all the mean reversion rate parameters and all the jump decay coefficient functions are identically equal to zero).

of the log of the value of the commodity, $\ln C_t \equiv \ln H(t,t)$. We can simplify this SDE and then substitute from equation 3.1, whence Ito's lemma gives equation 3.2. $\qquad\bullet$

Note that the SDE in equation 3.2 has a drift term which (by construction) is of a familiar form.

We can also use Ito's lemma applied to $\varepsilon(t,T) = -\partial \ln\big(H(t,T)P(t,T)/C_t\big)/\partial T$ to obtain the SDE for the dynamics of the instantaneous futures convenience yield forward rate $\varepsilon(t,T)$. Furthermore, if rewrite this SDE for $\varepsilon(t,T)$ for $\varepsilon(s,t)$ instead, and then rewrite this SDE in integral form from t_0 to t, then we can, with a little algebra, show that $\varepsilon(t,t) = \varepsilon_r(t)$ where $\varepsilon_r(t)$ is defined as in equation 3.1. This justifies our notation for $\varepsilon_r(t)$ and $\varepsilon(t,T)$ (ie it justifies our choice of $\varepsilon_r(t)$ in equation 3.1 and shows its consistency with equation 2.5). We will call $\varepsilon(t,t)$ (using terminology analogous to interest-rates) the futures convenience yield short rate.

Remark 3.2 : We have started the development of our model by assuming that the dynamics of futures commodity prices are as in equation 2.6 and then shown that our model implies that the dynamics of the value of the commodity and the futures convenience yield short rate are given by equations 3.2 and 3.1 respectively. One could, of course, go in the opposite direction and start by assuming the dynamics of equations 3.1 and 3.2 and then showing that the dynamics of futures commodity prices must be given by 2.6. However, it seems to us that starting with the dynamics of futures commodity prices is a much more natural starting point, firstly, because, one can directly observe the prices of, and trade in, futures contracts (which one certainly cannot do directly with convenience yields), and, secondly, the form of equation 2.6 seems much more intuitive than the form of equation 3.1.

Remark 3.3 : Note that the futures convenience yield short rate $\varepsilon(t,t)$ follows a mean reverting stochastic process and that it also (except in the special case that $b_m(t) \equiv 0$ for all m) exhibits jumps. This leads to an important conclusion. We know, from equation 2.6 that to capture the effect that long-dated futures contracts jump by smaller magnitudes than short-dated futures contracts, at least one of the M spot jump amplitudes must satisfy assumption 2.1 with $b_m(t) > 0$. If this is the case, then from equation 3.1, the futures convenience yield short rate $\varepsilon(t,t)$ must exhibit jumps. We now see why (see also remark 2.7) existing models in the literature such as Casassus and Collin-Dufresne (2005), Clewlow and Strickland (2000) and Hilliard and Reis (1998) **cannot capture the effect that long-dated futures contracts jump by smaller magnitudes than short-dated futures contracts. The existing models in the literature only consider jumps in spot commodity prices – they do not allow jumps in the dynamics of convenience yields**. In our notation, this is equivalent to assuming $b_m(t) \equiv 0$ for all m. Hence, existing models imply futures commodity prices across different tenors jump by the same proportional amounts which is contrary to the empirical evidence, for most commodities markets, presented in section 1.

Note that (observing equation 3.2) the volatility of the value of the commodity does not depend on the volatility of bond prices or interest-rates. If we examine the SDE for futures commodity prices (equation 2.6), the question might be asked: Why have the term $-\sigma_P(t,T)dz_P(t)$? In view of equation 2.5, Ito's lemma implies that the dynamics of $H(t,T)P(t,T) = C_t \exp\left(-\int_{s=t}^{T} \varepsilon(t,s)\,ds\right)$ do not depend on the term $-\sigma_P(t,T)dz_P(t)$. If we did not have the term $-\sigma_P(t,T)dz_P(t)$ in equation 2.6, then the dynamics of the value of the commodity and those of the futures convenience yield short rate would depend on the Brownian motion driving interest-rates and bond prices. Although there would be nothing wrong with this, it just seems less intuitively appealing. Of course, in a sense, writing the dynamics of futures commodity prices in the form of equation 2.6 is a non-assumption in that given the diffusion terms in equation 2.6 for any K, we can rewrite them in the form

$$\sum_{k=1}^{K} \sigma_{Hk}(t,T) dz_{Hk}(t) - \sigma_P(t,T) dz_P(t) = \sum_{k=1}^{K'} \sigma_{Hk}(t,T) dz_{Hk}(t), \text{ where } K' \equiv K+1,$$

$$dz_{HK'}(t) \equiv dz_P(t) \text{ and } \sigma_{HK'}(t,T) \equiv -\sigma_P(t,T) = -\frac{\sigma_r}{\alpha_r} + \frac{\sigma_r}{\alpha_r} \exp(-\alpha_r(T-t)).$$

Hence, the volatility term $\sigma_{HK'}(t,T)$ is still of the form of equation 2.7 and hence the model is still of essentially the same structure.

The following proposition provides further insight into our model because it shows that the log of the value of the commodity exhibits mean reversion.

Proposition 3.4 : The SDE for the log of the value of the commodity can be written:

$$d(\ln C_t) = a_{H1}(t)\big(\Lambda(t_0,t,\ln H(t_0,t)) - (\ln C_t)\big) dt + \sum_{k=1}^{K} \sigma_{Hk}(t,t) dz_{Hk}(t)$$

$$+ \sum_{m=1}^{M} \gamma_{mt} dN_{mt} - \left(\sum_{m=1}^{M} e_m(t,t) \right) dt \qquad (3.3)$$

where $\Lambda(t_0,t,\ln H(t_0,t))$ is a function of the state variables (whose exact form is easily obtained at the expense of some algebra).

Proof: We obtain an expression for $C_t \equiv H(t,t)$ by setting T equal to t in equation 2.12 and then we take logs. We then use this with the SDE for $\ln C_t \equiv \ln H(t,t)$, we obtained in the proof of proposition 3.1, to eliminate one of the state variables $X_{Hk}(t)$. The choice is arbitrary but to be definite, we eliminate $X_{H1}(t)$. We obtain equation 3.3. $\qquad \bullet$

Remark 3.5 : This shows that $\ln C_t$ follows a mean reverting jump-diffusion process with a long run mean reversion level of $\Lambda(t_0,t,\ln H(t_0,t))$. In fact, we can show that $\Lambda(t_0,t,\ln H(t_0,t))$ is itself also a mean reverting jump-diffusion process. Note also that if the jump decay coefficient functions $b_m(t)$ are strictly positive, then they also contribute to the effect of mean reversion. This can be seen even more clearly if we follow the proof of proposition 3.4 but, instead of eliminating the state variable $X_{H1}(t)$, we eliminate the jump state variable $X_{N1}(t)$, say. We then obtain an equation somewhat similar to equation 3.3 in form, but with the (mean reverting) drift term containing $b_1(t)$ instead of $a_{H1}(t)$. Indeed we see that the jump decay coefficient functions $b_m(t)$ play an analogous role for the jump processes as the mean reversion rate parameters $a_{Hk}(t)$ play for the Brownian motions. We noted, in section 1, the stylised empirical observation that, after a large jump in the price of a short-dated futures contract, the price often seems to revert to a more "usual" level very quickly. Our model is potentially able to capture this effect through large values of the jump decay coefficient functions.

How can we summarise this section? We have seen that the drift term in the SDE for the value of the commodity (equation 3.2) is equal to the risk-free rate minus the futures convenience yield short rate. The drift term (equation 3.3) can also simultaneously be viewed as that for a mean reverting stochastic process. The futures convenience yield short rate is a mean reverting stochastic process and (except in the special case that $b_m(t) \equiv 0$ for all m) it also exhibit jumps. Most importantly, we have shown that, in order to capture the effect that the prices of long-dated futures contracts jump by smaller magnitudes than short-dated contracts, it is necessary to have jumps in the dynamics of the futures convenience yield short rate – this is a feature which is missing from existing models in the literature.

4. Monte Carlo simulation

In this section, we show how we can simulate futures commodity prices. The key to this will be to simulate the state variables since then we can use equation 2.12. Monte Carlo simulation of the diffusion state variables is straightforward (see Babbs (1990), Dempster and Hutton (1997) or Glasserman (2004)). So now we examine how we can simulate the jump state variables, $X_{Nm}(t)$.

Firstly, for future notational convenience, we define, for each m,

$$\phi_m(t,T) \equiv \exp\left(-\int_t^T b_m(u)\,du\right). \tag{4.1}$$

Recall, that for each m, $m = 1,...,M$, N_{mt} has a Poisson distribution with intensity rate $\lambda_m(t)$. The process starts at zero ie $N_{mt_0} \equiv 0$ and every time a jump occurs, the process increments by one. Now, by the definition of a non-homogenous Poisson process, the probability $Q_m(t_0,t;n_m)$ that there are n_m jumps in the Poisson process N_{mt} in the time period t_0 to t is:

$$Q_m(t_0,t;n_m) \equiv \Pr(N_{mt} = n_m) = \exp\left(-\int_{t_0}^t \lambda_m(u)\,du\right)\frac{\left[\int_{t_0}^t \lambda_m(u)\,du\right]^{n_m}}{n_m!} \tag{4.2}$$

We now state a very useful mathematical proposition.

Proposition 4.1 : Suppose that we know that there have been n_m jumps between time t_0 and time t. Write the arrival times of the jumps as $S_{1m}, S_{2m},..., S_{n_m m}$. The conditional joint density function of the arrival times, when the arrival times are viewed as unordered random variables, conditional on $N_{mt} = n_m$ is:

$$\Pr\left(S_{1m} = s_{1m}\ \&\ S_{2m} = s_{2m}\ \&...\&\ S_{n_m m} = s_{n_m m}\mid N_{mt} = n_m\right) =$$

$$\left[\lambda_m(s_{1m})\right]\left[\lambda_m(s_{2m})\right]...\left[\lambda_m(s_{n_m m})\right]\Bigg/\left(\left[\int_{t_0}^t \lambda_m(u)\,du\right]^{n_m}\right) \tag{4.3}$$

Proof : The above result is proved in, for example, Karlin and Taylor (1975) in the case that the intensity rate is constant and the extension to a time-dependent deterministic intensity rate is relatively straightforward (and therefore the proof is omitted). •

This is an important result because now it is straightforward to simulate $X_{Nm}(t)$. Firstly, we simulate the number of jumps n_m up to time t. There are several ways to simulate the number of jumps, in a given time interval, of a non-homogenous Poisson process (for example, see Glasserman (2004)). Using equation 4.3, we can simulate the arrival times $S_{1m}, S_{2m},..., S_{n_m m}$ of the n_m jumps between time t_0 and time t. (This is particularly straightforward if $\lambda_m(t)$ is constant since then the arrival times, conditional on n_m, are uniform on (t_0,t). If $\lambda_m(t)$ is not constant, we can use the inverse transformation method (Glasserman (2004))).

Now note that equation 2.11, the definition of $X_{Nm}(t)$, implies that

$$X_{Nm}(t) = \sum_{i=1}^{n_m} \gamma_{mS_{im}} \exp\left(-\int_{S_{im}}^{t} b_m(u) \, du\right) = \sum_{i=1}^{n_m} \gamma_{mS_{im}} \phi_m(S_{im}, t).$$ (4.4)

If $n_m = 0$, then $X_{Nm}(t) = 0$. We include this case in equation 4.4 by using the usual convention that a summation is zero if the upper index is strictly less than the lower index.

It only remains to simulate γ_{mt} (in the case of assumption 2.1, the jump sizes are known constants, so, in fact, no further simulation is required, and, in the case of assumption 2.2, they are independent and identically distributed which means they do not depend on the arrival times, and, in fact, equation 4.4 simplifies since $b_m(t) \equiv 0$) and then we obtain $X_{Nm}(t)$ from equation 4.4.

In order to simulate futures commodity prices, we also need the final deterministic term in equation 2.12. This involves an integral which would, in general, have to be done numerically (see also Crosby (2006a)), but it is a simple one dimensional deterministic integral which can be pre-computed before entering the Monte Carlo simulation.

We will use the following proposition in section 5.

Proposition 4.2 :

$$E_{t_0}\left[\exp\left(\sum_{m=1}^{M}\left(\exp\left(-\int_{t}^{T} b_m(u)\,du\right)X_{Nm}(t)\right) - \sum_{m=1}^{M}\int_{t_0}^{t} e_m(s,T)\,ds\right)\right] = 1$$ (4.5)

Proof : Use equations 4.2 and 4.3 and standard results about conditional expectations.   ●

Remark 4.3 : Note (leaving aside the issue of any errors in the evaluation of the final deterministic integral in equation 2.12), that there are no discretisation error biases in the simulation of futures commodity prices in our model as there might be in some models involving the simulation of non-Gaussian stochastic processes (for discussions on this topic, see Babbs (2002) or Glasserman (2004)).

5. Option pricing

Our aim in this section is to derive the prices of standard options and to do so (in spite of the obvious fact that, in our model, futures commodity prices are not log-normally distributed) in a form suitable for rapid computation. The key to this will be the observation that, **conditional** on the number of jumps and their arrival times (and with a suitable assumption about the spot jump amplitudes), futures commodity prices **are** log-normally distributed, at which point familiar results come into play (see also Merton (1976) and Jarrow and Madan (1995)). We will derive the prices of standard European options on futures, futures-style options on futures and standard European options on forward commodity prices. Later in this section, we will provide some numerical examples which illustrate our model. We will also show that we can rapidly (typically of the order of 1/50th of a second per option depending upon the required accuracy) compute the prices of standard options.

To achieve our goals, we will have to make an assumption about the distribution of the spot jump amplitudes γ_{mt} in the case of assumption 2.2. We assume the spot jump amplitudes γ_{mt} are normally distributed.

For each m :

In the case of assumption 2.1, the spot jump amplitudes are assumed to be equal to β_m, a constant.

In the case of assumption 2.2, the spot jump amplitudes are assumed to be normally distributed with mean β_m and standard deviation υ_m, which then implies $\exp\left(\gamma_{ms}\exp\left(-\int_{s}^{T} b_m(u)\,du\right)\right)$ is log-normally distributed and using standard results, we have (putting $b_m(t) \equiv 0$)

$$E_{Nms}\left(\exp\left(\gamma_{ms}\exp\left(-\int_s^T b_m(u)\,du\right)\right)\right) = \exp\left(\beta_m + \frac{1}{2}\upsilon_m^2\right) \tag{5.1}$$

A generic option pricing formula:

Our aim is to value, at time t, a European (non-path-dependent) option, maturing at time T_1, written on the futures commodity price, where the futures contract matures at time T_2, $(T_2 \geq T_1 \geq t)$.

Firstly, define the indicator functions, for each m, $m = 1,...,M$,

$1_{m(2.1)} = 1$ if assumption 2.1 is satisfied, for this m, and $1_{m(2.1)} = 0$ otherwise

and $1_{m(2.2)} = 1$ if assumption 2.2 is satisfied, for this m, and $1_{m(2.2)} = 0$ otherwise.

Conditional on the number of jumps n_m, $m = 1,...,M$, in the time period t to T_1, and the arrival times $s_{1m}, s_{2m},..., s_{n_m m}$, $m = 1,...,M$ of these jumps, then (using equations 4.1 and 4.4):

$$\exp\left(\left(\exp\left(-\int_{T_1}^{T_2} b_m(u)\,du\right)X_{Nm}(T_1)\right)\right) = \exp\left(\sum_{i=1}^{n_m}\beta_m\phi(s_{im},T_2)\right) \text{ if } 1_{m(2.1)} = 1 \tag{5.2}$$

or:

$$\exp\left(\left(\exp\left(-\int_{T_1}^{T_2} b_m(u)\,du\right)X_{Nm}(T_1)\right)\right) \text{ is log-normally distributed with mean}$$

$$\exp\left(\sum_{i=1}^{n_m}\left(\beta_m + \frac{1}{2}\upsilon_m^2\right)\right) = \exp\left(n_m\left(\beta_m + \frac{1}{2}\upsilon_m^2\right)\right) \text{ if } 1_{m(2.2)} = 1 \tag{5.3}$$

Proposition 5.1 : The futures commodity price $H(T_1,T_2)$ at time T_1 to time T_2, conditional on the futures price $H(t,T_2)$ at time t (where $t \leq T_1 \leq T_2$) and conditional on the number of jumps n_m, $m = 1,...,M$, in the time period t to T_1, and the arrival times $s_{1m}, s_{2m},..., s_{n_m m}$, $m = 1,...,M$ of these jumps, is log-normally distributed with mean

$$H(t,T_2)\exp\left(\left(\sum_{m=1}^{M}\left\{1_{m(2.1)}\sum_{i=1}^{n_m}\beta_m\phi(s_{im},T_2)+1_{m(2.2)}n_m\left(\beta_m + \frac{1}{2}\upsilon_m^2\right)\right\}\right)-\left(\sum_{m=1}^{M}\int_t^{T_1}e_m(s,T_2)\,ds\right)\right)$$

$$\equiv H(t,T_2)V(t,T_1;n_m;T_2,M) \tag{5.4}$$

where $V(t,T_1;n_m;T_2,M)$ is defined by equation 5.4 and where:

$$\exp\left(-\int_t^{T_1}e_m(s,T_2)\,ds\right) = \exp\left(-\int_t^{T_1}\lambda_m(s)\left(\exp(\phi_m(s,T_2)\beta_m)-1\right)ds\right) \text{ if } 1_{m(2.1)} = 1 \tag{5.5}$$

and

$$\exp\left(-\int_t^{T_1}e_m(s,T_2)\,ds\right) = \exp\left(-\left\{\exp\left(\beta_m + \frac{1}{2}\upsilon_m^2\right)-1\right\}\left[\int_t^{T_1}\lambda_m(u)\,du\right]\right) \text{ if } 1_{m(2.2)} = 1 \tag{5.6}$$

15

Proof: Equation 5.4 follows immediately from equation 2.12, taken together with equations 5.2 and 5.3. Equations 5.5 and 5.6 use the definitions in equations 4.1 and 2.8. Note that, in general, it would be necessary to compute the integral in equation 5.5 numerically. $\bullet$

Now, define $\Sigma^2\left(t,T_1,T_2,M\right)$, via:

$$\Sigma^2\left(t,T_1,T_2,M\right) \equiv \int_t^{T_1}\left(\sum_{k=1}^K \sigma_{Hk}^2\left(s,T_2\right)+\sigma_P^2\left(s,T_2\right)-2\sum_{k=1}^K \rho_{PHk}\sigma_P\left(s,T_2\right)\sigma_{Hk}\left(s,T_2\right)\right)ds$$

$$+\int_t^{T_1}\left(\sum_{k=1}^K\sum_{j=1}^{k-1}2\rho_{HkHj}\sigma_{Hk}\left(s,T_2\right)\sigma_{Hj}\left(s,T_2\right)\right)ds+\sum_{m=1}^M\left(n_m\upsilon_m^2 1_{m(2.2)}\right) \qquad (5.7)$$

Consider a non-path-dependent European option written on the futures commodity price. The option matures at time T_1 and the futures contract matures at time T_2. Let the payoff of the option at time T_1 be $D\left(H\left(T_1,T_2\right)\right)$ for some $\mathfrak{S}_{T_1}$-measurable function $D(\bullet)$.

Conditional on the number of jumps n_m, $m=1,...,M$ and the arrival times $s_{1m},s_{2m},...,s_{n_m m}$, $m=1,...,M$ of these jumps, the value of the option at time t is (where $t\le T_1\le T_2$):

$$E_t\left[\exp\left(-\int_t^{T_1}r(u)du\right)D\left(H\left(T_1,T_2\right)\right)|\,n_m,s_{1m},s_{2m},...,s_{n_m m};m=1,...,M\right] \qquad (5.8)$$

Remark 5.2 : In view of proposition 5.1, given the payoff $D\left(H\left(T_1,T_2\right)\right)$ of the option at time T_1, we will be able to use standard results (for log-normally distributed prices), together with equations 5.4, 5.5, 5.6 and 5.7, to calculate the expectation in equation 5.8.

Note that the probability that there are n_m jumps in the Poisson process N_{mt} in the time period t to T_1, for each m, $m=1,...,M$, is $Q_m\left(t,T_1;n_m\right)$, where $Q_m\left(t,T_1;n_m\right)$ is defined as in equation 4.2.

Proposition 5.3 : The price of the option at time t is:

$$\sum_{n_1=0}^{n_1=\infty}\sum_{n_2=0}^{n_2=\infty}...\sum_{n_M=0}^{n_M=\infty}Q_1\left(t,T_1;n_1\right)Q_2\left(t,T_1;n_2\right)...Q_M\left(t,T_1;n_M\right)$$

$$\int_t^{T_1}\int_t^{T_1}...\int_t^{T_1}E_t\left[\exp\left(-\int_t^{T_1}r(u)du\right)D\left(H\left(T_1,T_2\right)\right)|\,n_m,s_{1m},s_{2m},...,s_{n_m m};m=1,...,M\right]$$

$$\prod_{m=1}^M\frac{\left[\lambda_m\left(s_{1m}\right)\right]\left[\lambda_m\left(s_{2m}\right)\right]...\left[\lambda_m\left(s_{n_m m}\right)\right]}{\left[\int_t^{T_1}\lambda_m(u)du\right]^{n_m}}ds_{11}ds_{21}...ds_{n_1 1}ds_{12}ds_{22}...ds_{n_2 2}...ds_{1M}ds_{2M}...ds_{n_M M}$$

$$(5.9)$$

Proof: It follows immediately from the results of section 4 (in particular equations 4.2 and 4.3) and standard results about conditional expectations. $\bullet$

Remark 5.4 : Further, in the special case that assumption 2.2 is satisfied for all m, $m=1,...,M$, the form of $V\left(t,T_1;n_m;T_2,M\right)$ is simplified and the option price at time t also simplifies to:

$$\sum_{n_1=0}^{n_1=\infty} \sum_{n_2=0}^{n_2=\infty} ... \sum_{n_M=0}^{n_M=\infty} Q_1\left(t,T_1;n_1\right)Q_2\left(t,T_1;n_2\right)...Q_M\left(t,T_1;n_M\right)$$

$$E_t\left[\exp\left(-\int_t^{T_1} r(u)\,du\right)D\left(H\left(T_1,T_2\right)\right)\mid n_m;m=1,...,M\right] \qquad (5.10)$$

Using equations 5.4, 5.7, 5.8 and 5.9, we are now in a position to write down the prices of various standard options. Our specific option pricing formulae will come from substituting a specific form for equation 5.8 into[8] equation 5.9. We state the results without proof but for full details and methodologies, see Merton (1973),(1976), Babbs (1990), Amin and Jarrow (1991), Duffie and Stanton (1992), Jamshidian (1993), Jarrow and Madan (1995) and especially Miltersen and Schwartz (1998). In each of the following option pricing formulae (equations 5.11 to 5.14), we denote the strike of the option by K and we write $\eta = 1$ if the option is a call and $\eta = -1$ if the option is a put.

Standard European Options on Futures:

Suppose that we wish to value at time t a standard European (call or put) option on the futures commodity price. The option matures at time T_1 and the futures contract matures at time T_2, where $T_2 \geq T_1$. The payoff of the option at time T_1 is $\max\left(\eta\left(H\left(T_1,T_2\right)-K\right),0\right)$.

The price of the option at time t is given by[8] equation 5.9 with

$$E_t\left[\exp\left(-\int_t^{T_1} r(u)\,du\right)D\left(H\left(T_1,T_2\right)\right)\mid n_m,s_{1m},s_{2m},...,s_{n_mm};m=1,...,M\right] \text{ replaced by}$$

$$\eta P\left(t,T_1\right)\left(H\left(t,T_2\right)V\left(t,T_1;n_m;T_2,M\right)\exp\left(\int_t^{T_1} A\left(s,T_1,T_2\right)ds\right)N\left(\eta d_1\right)-KN\left(\eta d_2\right)\right)$$

$$(5.11)$$

where

$$A\left(s,T_1,T_2\right)\equiv\left(\sum_{k=1}^K \rho_{PHk}\sigma_P\left(s,T_1\right)\sigma_{Hk}\left(s,T_2\right)\right)-\sigma_P\left(s,T_1\right)\sigma_P\left(s,T_2\right),$$

$$d_1 \equiv \frac{\ln\left(H\left(t,T_2\right)V\left(t,T_1;n_m;T_2,M\right)/K\right)+\left(\int_t^{T_1} A\left(s,T_1,T_2\right)ds\right)+\frac{1}{2}\Sigma^2\left(t,T_1,T_2,M\right)}{\Sigma\left(t,T_1,T_2,M\right)},$$

$d_2 \equiv d_1 - \Sigma\left(t,T_1,T_2,M\right)$ and $V\left(t,T_1;n_m;T_2,M\right)$ is as in equation 5.4.

Futures-style Options on Futures:

Suppose that we wish to value, at time t, a futures-style option (call or put) on the futures commodity price. Futures-style options are traded on some exchanges and the key point about them is that they are similar to futures contracts in that the gains and losses of the futures-style option are resettled continuously (in practice, daily), with a mark-to-market procedure, and, as with futures contracts, there is no initial cost in buying a futures-style option. We assume that the futures-style option matures at time T_1 and the futures contract matures at time T_2, where $T_2 \geq T_1$. The futures-style option price (ie its delivery value) at time T_1 is $\max\left(\eta\left(H\left(T_1,T_2\right)-K\right),0\right)$. It can be shown (see Merton (1990), Duffie (1996) or Duffie and Stanton (1992)) that the futures-style option price, at time t, is the price of a standard (ie non-futures-style) option, at time t, which has a payoff of

$$\left\{ \exp\left(\int_t^{T_1} r(u)\,du \right) \left[\max\left(\eta\left(H(T_1, T_2) - K \right), 0 \right) \right] \right\}$$ at time T_1. Hence the futures-style option price, at time t, is:

$$E_t \left[\exp\left(-\int_t^{T_1} r(u)\,du \right) \exp\left(\int_t^{T_1} r(u)\,du \right) \left[\max\left(\eta\left(H(T_1, T_2) - K \right), 0 \right) \right] \right]$$

$$= E_t \left[\max\left(\eta\left(H(T_1, T_2) - K \right), 0 \right) \right]$$

Hence, we can show that the futures-style option price at time t is given by[8] equation 5.9 with equation 5.8 replaced by

$$\eta\left(H(t, T_2) V(t, T_1; n_m; T_2, M) N(\eta d_1) - K N(\eta d_2) \right) \qquad (5.12)$$

where

$$d_1 \equiv \frac{\ln\left(H(t, T_2) V(t, T_1; n_m; T_2, M)/K \right) + \frac{1}{2}\Sigma^2(t, T_1, T_2, M)}{\Sigma(t, T_1, T_2, M)} \text{ and } d_2 \equiv d_1 - \Sigma(t, T_1, T_2, M)$$

Remark 5.5 : It can be shown (using the methods of Merton (1973), Duffie and Stanton (1992) and Jamshidian (1993)) that it is never optimal to exercise American futures-style options on futures prices before maturity (this applies to both calls and puts). Hence equation 5.12, combined with[8] equation 5.9, is equally valid for both European and American futures-style options on futures prices.

Standard European Options on Forwards:

Suppose that we wish to value at time t a standard European (call or put) option on the forward commodity price. The option matures at time T_1 and the forward price is to time T_2, where $T_2 \geq T_1$.

There are two possible payoffs:

We consider first the case where the payoff of the option at time T_1 is $\max\left(\eta\left(F(T_1, T_2) - K \right), 0 \right)$. The price of the option at time t is given by[8] equation 5.9 with equation 5.8 replaced by

$$\eta P(t, T_1) \left(F(t, T_2) V(t, T_1; n_m; T_2, M) \exp\left(\int_t^{T_1} B(s, T_1, T_2)\,ds \right) N(\eta d_1) - K N(\eta d_2) \right)$$

$$(5.13)$$

where

$$B(s, T_1, T_2) \equiv \left(\sum_{k=1}^{K} \rho_{PHk} \left(\sigma_P(s, T_1) - \sigma_P(s, T_2) \right) \sigma_{Hk}(s, T_2) \right)$$

$$- \left(\sigma_P(s, T_1) - \sigma_P(s, T_2) \right) \sigma_P(s, T_2),$$

$$d_1 \equiv \frac{\ln\left(F(t, T_2) V(t, T_1; n_m; T_2, M)/K \right) + \left(\int_t^{T_1} B(s, T_1, T_2)\,ds \right) + \frac{1}{2}\Sigma^2(t, T_1, T_2, M)}{\Sigma(t, T_1, T_2, M)}$$

and $d_2 \equiv d_1 - \Sigma(t, T_1, T_2, M)$

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[8] or equation 5.10, in the special case noted in remark 5.4.

We consider secondly the case where the payoff of the option is also $\max\left(\eta\left(F\left(T_1,T_2\right)-K\right),0\right)$ but now the payoff occurs at time T_2. This means the payoff is the same as a payoff of $\max\left(\eta P\left(T_1,T_2\right)\left(F\left(T_1,T_2\right)-K\right),0\right)$ at time T_1.

The price of the option at time t is given by[8] equation 5.9 with equation 5.8 replaced by

$$\eta P\left(t,T_2\right)\left(F\left(t,T_2\right)V\left(t,T_1;n_m;T_2,M\right)N\left(\eta d_1\right)-KN\left(\eta d_2\right)\right) \tag{5.14}$$

where

$$d_1 \equiv \frac{\ln\left(F\left(t,T_2\right)V\left(t,T_1;n_m;T_2,M\right)\big/K\right)+\frac{1}{2}\Sigma^2\left(t,T_1,T_2,M\right)}{\Sigma\left(t,T_1,T_2,M\right)} \text{ and } d_2 \equiv d_1 - \Sigma\left(t,T_1,T_2,M\right)$$

As an immediate corollary, we can obtain the price of a standard European option on the spot commodity price, by setting T_2 equal to T_1 in equation 5.13 or 5.14 (substituted into[8] equation 5.9).

Numerical examples and computational issues:

The above results are very useful as they also allow the possibility to calibrate the model through deriving implied parameters from the market prices of options (for which purpose rapid computation is important). We will, later in this section, illustrate our model with a total of five numerical examples, which we split into two categories, labelled examples 1 to 3 and examples 4 and 5.

Firstly, we discuss computational issues surrounding the rapid computation of option prices using equations 5.9 to 5.14. The probabilities in the Poisson mass functions will rapidly tend to zero once the number of jumps is greater than the mean number of jumps. Therefore, computation times in the case when all the Poisson processes satisfy assumption 2.2 will typically be very small (at least when M is not too large). When all or some of the Poisson processes satisfy assumption 2.1, it is necessary to compute the integrals over the arrival times. The most appropriate method would seem to be to use Monte Carlo simulation of the arrival times (we stress only of the arrival times – not of the number of Poisson jumps nor the diffusion processes which can be done analytically). This is the method we use in the numerical examples below. Although this might sound computationally intensive, the simulation is just of the arrival times of the jumps. In many cases, the variation of the integral with different arrival times will be quite small leading to small standard errors. This might typically be the case for options which are deep in or out of the money or when the jump decay coefficient functions ($b_m\left(t\right)$) are close to zero. In addition to minimise standard errors, we used the method of antithetic variates and we also used equation 4.5 as a control variate using the optimal-weighting/linear-regression methodology described, for example, in Glasserman (2004). The option prices in example 2 and (when appropriate) in examples 4 and 5 were all computed using 1500 Monte Carlo simulations.

The deterministic integral in equation 5.5 was computed using the trapezium rule with 2500 points. Using a much larger number of points confirmed that the potential errors in the option prices in examples 2, 4 and 5 due to the approximation inherent in computing this integral were, in all cases, much less than 0.000001 which is negligible compared to the standard errors reported. In all the examples, the summation over the Poisson mass functions was truncated when both the proportional and absolute convergence of the option price were less than 0.0001. Computations were performed on a desk-top p.c., running at 2.8 GHz, with 1 Gb of RAM with a program written in Microsoft C++.

We now illustrate our model with our first three examples, labelled examples 1 to 3, the results of which are in tables 1 to 3 respectively.

In all three examples 1 to 3, we assume that the futures commodity prices to all maturities are 95. We assume that interest-rates are stochastic and that $\sigma_r = 0.0096$ and $\alpha_r = 0.2$. We assume the interest-rate yield curve is flat with a continuously compounded risk-free rate of 0.05 (as in Miltersen and Schwartz (1998)).

Although many of the parameters in our model can be time-dependent (and indeed it may be useful to allow for this to capture, for example, seasonality (see Miltersen (2003))), we will illustrate the model with constant parameters. In order to match the parameters of Miltersen and Schwartz (1998), whose set-up is slightly different to ours but entirely equivalent in the two factor pure-diffusion case,

we choose to have two Brownian motions (in addition to the Brownian motion driving interest-rates) ie $K = 2$ and $\eta_{H1} = 0.266$, $\eta_{H2} = 0.249/1.045 \approx 0.23827751196$, $\chi_{H1} = 0.0$, $\chi_{H2} = -0.249/1.045$, $a_{H2} = 1.045$, $\rho_{H1H2} = -0.805$, $\rho_{PH1} = -0.0964$, $\rho_{PH2} = 0.1243$.

Note the negative value of χ_{H2} is artificial in order to match the Miltersen and Schwartz (1998) data and could be made positive by combining η_{H1} and η_{H2} into one term and making consistent adjustments to the correlations in the obvious manner.

We now consider examples 1 to 3. Example 1 is pure-diffusion and examples 2 and 3 are with jumps. The pure-diffusion example is effectively identical to that used in Miltersen and Schwartz (1998). We value standard European call options (using equations 5.9 and 5.11) on futures contracts whose maturities are 0.125 years after the maturity of the option. We price options with strikes 75, 80, 95, 110, 115 and maturities equal to 0.25, 0.5, 0.75, 1, 2, 3 years (there are 30 options in total).

Example 1 :
In example 1, we price options in the pure-diffusion case (using equation 5.11). The results are in table 1. Clearly the results are exactly as in table 1 of Miltersen and Schwartz (1998) (we have extra option maturities and extra strikes) since we have (albeit in a slightly different form) the same diffusion parameters.   ●

Now we introduce jump processes for examples 2 and 3 but keep the diffusion parameters as in example 1. The parameters of our processes are purely for illustration.

Example 2 :
In example 2, we assume that there is one Poisson process, $M = 1$ and it satisfies assumption 2.1 and it has constant parameters:
$$\lambda_1 = 0.75, \ \beta_1 = 0.22, \ b_1 = 2.0$$

The parameters are only for illustration. The value of b_1 is roughly equivalent to the effect of a jump being "dampened" to approximately 37.8 % of the jump size over half a year which seems plausible.
Now we price options, using equations 5.9 and 5.11, with all the other parameters the same as in example 1. The results are in table 2. Also in the table are the corresponding standard errors (all are less than 0.0028) and the corresponding implied Black (1976) volatilities with[9] a price of 95.
The total computation time for all 30 options in this example was less than 0.51 seconds – or an average of less than 0.017 seconds per option.   ●

Example 3 :
In example 3, we assume that there are two Poisson processes, $M = 2$ and they both satisfy assumption 2.2, with parameters:
$$\lambda_1 = 0.75, \ \beta_1 = 0.22, \ \upsilon_1 = 0.01 \text{ (and } b_1 = 0.0 \text{ which is required for assumption 2.2)}$$
$$\lambda_2 = 0.75, \ \beta_2 = -0.15, \ \upsilon_2 = 0.01 \text{ (and } b_2 = 0.0 \text{ ditto)}$$

Again, the parameters are only for illustration. The intuition of the parameter values, loosely speaking, is to try to capture a commodity which can have upward jumps and also have downward jumps of slightly smaller size, the intensity rates of the two Poisson processes being equal.
Now we again price options, using equations 5.10 and 5.11, with the other parameters the same as in example 1. The results are in table 3. Also in the table are the corresponding implied Black (1976) volatilities with[9] a price of 95. Since $b_1 = 0.0$ and $b_2 = 0.0$, there is no integration over the arrival times and hence computation times were negligible compared to those in example 2.   ●

It can be seen that in both examples with jumps (that is, examples 2 and 3), the model produces a volatility skew. The magnitude of the skew decreases with increasing option maturity which is typical for jump-diffusion processes (and is also in line with empirical observations in the commodity options

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[9] We use the futures price of 95 when calculating the implied Black (1976) volatilities for all our examples because this seems to be in line with the market convention even though this convention appears to effectively ignore the impact of stochastic interest-rates and hence the difference between forward prices and futures prices.

markets (Geman (2005)). In all three examples, we see that implied volatilities decrease with increasing option maturity (again in line with typical empirical observations (Geman (2005)).

Calibration to market data:

We will now calibrate two different specifications of our model to market data. As in other options markets, the commodity options market quotes implied Black (1976) volatilities. We obtained from inter-dealer brokers live market data, as of 25th January 2005, for the implied Black (1976) volatilities of OTC options on crude oil futures. The options had eleven different maturities, namely, 1 month, 2 months, 3 months, 6 months, 12 months, 18 months, 2 years, 3 years, 4 years, 5 years and 6 years. The options, at each maturity, were of seven different strikes. The strikes were different for each maturity because the market convention is to quote options at strikes given by specific Black (1976) deltas. In other words, the strike is determined by solving for the strike that gives a specified delta when substituted into the formula for the delta in the Black (1976) model. The options are either puts or calls according to which is out-of-the-money (except, obviously, for at-the-money-forward options when puts and calls have the same price). Our options were at the strikes corresponding to (in order of increasing strike price) deltas of -0.1 (put), -0.25 (put), -0.35 (put), at-the-money-forward, 0.35 (call), 0.25 (call) and 0.1 (call). Thus we have seven different strikes at each of eleven option maturities ie a total of 77 options. For all eleven different option maturities, the futures contract maturities are between 11 and 15 days after option maturity (the average time between futures contract maturity and the option maturity was 13 days but the exact time depends on holidays and the dates of weekends).

We first extracted the US dollar interest-rate yield curve from LIBOR deposit rates and by bootstrapping swap rates. We then determined the extended Vasicek model parameters by calibrating the extended Vasicek (Babbs (1990), Hull and White (1993)) model to the market prices of liquid European swaptions. We then calibrated two different specifications of our model to the market implied Black (1976) volatilities. The specifications were as follows:

In each specification, we had two Brownian motions (in addition to the one driving interest-rates). We set $\eta_{H2} \equiv 0$ (in order to avoid a degeneracy) and we assumed η_{H1}, χ_{H1}, χ_{H2}, a_{H1} and a_{H2} were all constants. In each specification, we had two Poisson processes ie $M = 2$.

In the first specification (we will call it Specification 1), the jumps amplitudes were both of the type of assumption 2.1, with λ_1, λ_2, b_1, b_2 all assumed constants and $b_1 > 0$, $b_2 > 0$. The parameters to be determined from the calibration were thus η_{H1}, χ_{H1}, χ_{H2}, a_{H1}, a_{H2}, ρ_{H1H2}, ρ_{PH1}, ρ_{PH2}, λ_1, λ_2, β_1, β_2, b_1, b_2. Therefore, there were 14 parameters (in addition to σ_r and α_r which were calibrated independently to the market prices of European swaptions).

However, in the second specification (we will call it Specification 2), the jump amplitudes were both of the type of assumption 2.2, with λ_1, λ_2 assumed constants. The parameters to be determined from the calibration were thus η_{H1}, χ_{H1}, χ_{H2}, a_{H1}, a_{H2}, ρ_{H1H2}, ρ_{PH1}, ρ_{PH2}, λ_1, λ_2, β_1, β_2, υ_1, υ_2. Therefore, there were 14 parameters – the same as for specification 1.

To be precise, in our calibration, we solved for the parameters which minimised the sums of squares of proportional differences between the model and market prices of the 77 options. The results of the calibration were that we obtained model parameters as follows:

Extended Vasicek parameters: $\sigma_r = 0.0109$, $\alpha_r = 0.0403$.

Specification 1 parameters:

$\eta_{H1} = 0.1646$, $\eta_{H2} \equiv 0$ (by design), $\chi_{H1} = 0.2293$, $\chi_{H2} = 0.0795$, $a_{H1} = 1.6407$,

$a_{H2} = 0.0603$, $\rho_{H1H2} = -0.4134$, $\rho_{PH1} = -0.3485$, $\rho_{PH2} = -0.3562$,

$\lambda_1 = 0.7114$, $\lambda_2 = 0.1600$, $\beta_1 = -0.2427$, $\beta_2 = 0.2509$, $b_1 = 0.7189$, $b_2 = 1.0280$

Specification 2 parameters:

$\eta_{H1} = 0.1034$, $\eta_{H2} \equiv 0$ (by design), $\chi_{H1} = 0.3271$, $\chi_{H2} = 0.0577$, $a_{H1} = 1.5781$,

$a_{H2} = 0.1088$, $\rho_{H1H2} = -0.3743$, $\rho_{PH1} = -0.3280$, $\rho_{PH2} = -0.3451$,

$\lambda_1 = 0.6717$, $\lambda_2 = 0.0588$, $\beta_1 = -0.1580$, $\beta_2 = 0.1743$, $\upsilon_1 = 0.0759$, $\upsilon_2 = 0.0199$

We have plotted the results of the calibration, graphically, in terms of implied Black (1976) volatilities, in figures 1 and 2 (only two maturities, namely 1 month and 4 years, are shown for brevity but the rest of the data is qualitatively similar). Both specifications were able to give a good fit to the market implied Black (1976) volatilities. However, the fit for specification 1 is much better than that for specification 2. The residual value of the sum of the squares of the proportional differences between the model and market prices was 0.2181 for specification 1 whereas it was 0.3778 for specification 2. In addition, the sum of the squares of the differences between the model and market implied volatilities was 0.00223 for specification 1 whereas it was 0.00467 for specification 2. In addition, the maximum difference (in absolute value), across all 77 options between the model and market implied volatilities was 1.268 percentage points for specification 1 whereas it was 1.980 percentage points for specification 2. Thus, by three different measures, specification 1 gave a much better fit to the market data than specification 2. In addition, we note that if we exclude the options with -0.1 delta and 0.1 delta (ie in the wings), the differences (in absolute value) between the model and the market implied Black (1976) volatilities, for specification 1, for the remaining 55 options, were all under one percentage point. This is indicative of an excellent fit because 1% is the approximate bid-offer spread (in volatility terms) for these options (the bid-offer spread would typically be somewhat wider for the options with -0.1 delta and 0.1 delta). To summarise, we have seen the fit to market data is much improved when the jumps both satisfy assumption 2.1 (as in specification 1) which is in line with the empirical features of the commodities markets presented in section 1.

We will now provide a further two examples, labelled examples 4 and 5, which illustrate our model using the model parameters we have calibrated above.

In both examples 4 and 5, we price standard European call options with an option maturity of 2 years. The (continuously compounded) risk free spot interest-rate to two years is 0.03579. Hence $T_1 = 2$ and $P(t, T_1) \approx 0.930921801$. In each example, we price the options with the calibrated data from above for each of the two specifications of jumps processes.

Example 4 :
In example 4, we price standard European call options on futures contracts maturing 13 days after the option maturity. Options with an option maturity of two years and a futures contract maturity 13 days later formed a subset of the options we used in the calibration to the market prices of options on crude oil futures above (but now we will consider slightly different strikes). The futures price is 41.02. Hence $T_1 = 2$, $T_2 = 2 + (13/365) \approx 2.035616438$ and $H(t, T_2) = 41.02$. We priced standard European calls whose strikes are 37.02, 41.02 and 45.02 ie with strikes which are the current futures price and 4 dollars either side. The results are in table 4 where we display the option price, the implied Black (1976) volatility and (for specification 1) the standard error of the option price. •

Example 5 :
In example 5, we price standard European call options on futures contracts maturing 3 years and 13 days after the option maturity (the option maturity is, again, two years). In the calibration to the market prices of options on crude oil futures above, 5 year options into these same futures contracts formed a subset of the options to which we calibrated. By contrast, in this example, we are pricing 2 year options into these futures contracts ie we are now pricing 2 year options into futures contracts which mature 3 years and 13 days after the option maturity. The futures price is 28.42. Hence $T_1 = 2$, $T_2 = 2 + 3 + (13/365) \approx 5.035616438$, $H(t, T_2) = 28.42$. We priced standard European calls whose strikes are 24.42, 28.42 and 32.42 ie with strikes which are the current futures price and 4 dollars either side. The results are in table 5 where we display the option price, the implied Black (1976) volatility and (for specification 1) the standard error of the option price. •

Examining table 4 closely we see that the prices of the three options are very, very close in specification 1 and specification 2. This is not surprising since options with the same option maturity and the same futures contract maturity (albeit with slightly different strikes) were used in the calibration. The option prices in specification 1 are just very slightly higher than for specification 2, for each of the three strikes, but the maximum difference between the implied volatilities of the corresponding three options in the two different specifications is 0.3 %.

Examining table 5 closely we see a different picture. For each of the three options, the option price is now rather lower for specification 1 than for specification 2. This seems intuitive because in specification 1, we have jumps whose magnitudes are exponentially dampened with futures contract tenor. We also see that the implied volatility of the three options for specification 1 imply a much

flatter skew than in specification 2. Furthermore, whilst the slope of the skew in specification 1 is very slightly upwards (ie the implied volatility of the three call options slightly increases with increasing strike), the slope of the skew in specification 2 is downwards (ie the implied volatility of the three call options decreases with increasing strike).

It is well-known that it is possible to have two different stochastic processes which give rise to the same prices for standard European options but give rise to different prices for exotic options. In examples 4 and 5, we are seeing the same issue at work.

We cannot reject or accept either specification 1 or specification 2 on the basis of the option prices in examples 4 and 5 but we note again that specification 1 is in line with the empirical observations we presented in section 1 and it also gives a better fit to the market prices of options than specification 2.

Miltersen and Schwartz (1998) remark (and we concur) on the importance of futures maturity (not just option maturity) on option prices. Our experience is that, at the moment, most standard (plain vanilla) options currently traded in the commodities markets have maturities which are usually less than two months (and often just a few weeks) before the maturity of the underlying futures contracts. As the commodity derivative markets expand, this may well change (compare the development of the interest-rate derivatives markets: at one time, caps were much more common than swaptions but now the situation is almost reversed), in which case, as examples 4 and 5 show, our model will be particularly useful.

Our model can also price complex (exotic) commodity derivatives utilising the Monte Carlo simulation method described in section 4.

A possible area for future research would be to further investigate variance reduction techniques or to use quasi-random number methods (see for example, Glasserman (2004)) to speed up the evaluation of option prices. A further possibility is to price options using the Fourier Transform methodology of, for example, Carr and Madan (1999) and this is the subject of further research, on which we report in Crosby (2006a). Finally, extending our model to allow for a multi-factor (rather than one factor) Gaussian interest-rate model is very straightforward.

6. Conclusions

We have considered a tractable arbitrage-free multi-factor jump-diffusion model for the evolution of futures commodity prices consistent with any initial term structure. The model allows long-dated futures contracts to jump by smaller magnitudes than short-dated futures contracts, which is in line with stylised empirical observations (especially for energy-related commodities). This is a feature which, to our best knowledge has never appeared in the literature before. We have related this model to forward commodity prices and to the value of the commodity. We have shown that the value of the commodity exhibits mean reversion in the risk-neutral equivalent martingale measure. We have related our model to stochastic convenience yields which themselves exhibit mean reversion and also, in general (but depending on the precise specification), exhibit jumps. Whilst some of the expressions appear quite long, the model described in this paper is conceptually very intuitive. The model is highly amenable to Monte Carlo simulation (without discretisation error bias), utilising equations derived in section 4. Hence, Monte Carlo simulation can be used to price complex (exotic) commodity derivatives within our model. We have demonstrated that our model can produce volatility skews which, in line with those observed in the commodity options markets, are much more pronounced for options with shorter maturities. We have shown that the prices of standard options have semi-analytical solutions which can be rapidly evaluated (in typically of the order of $1/50^{th}$ of a second). This opens the possibility of calibrating the model through deriving implied parameters from the market prices of options. We have calibrated our model to the market prices of options on crude oil futures and shown that, by allowing the feature that the prices of long-dated futures contracts can jump by smaller magnitudes than short-dated futures contracts, we can get a significantly better fit.
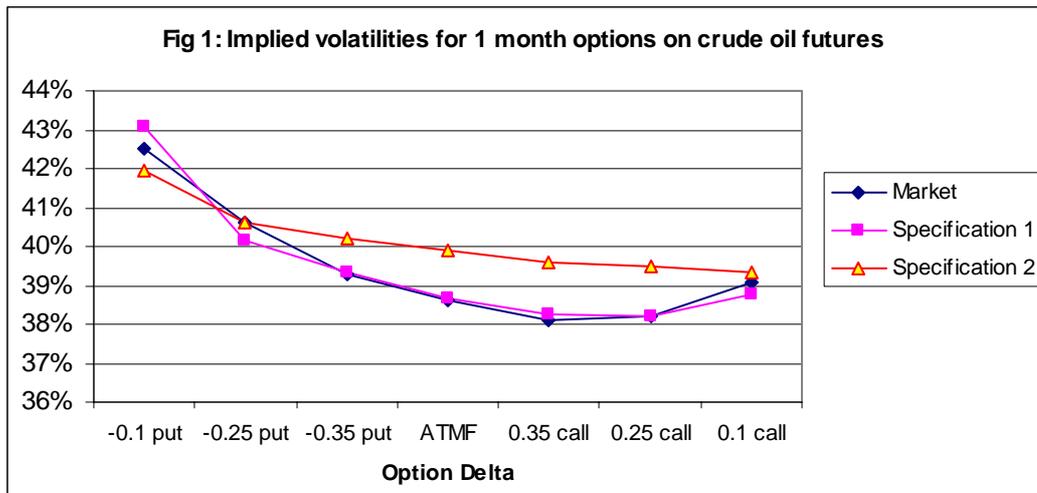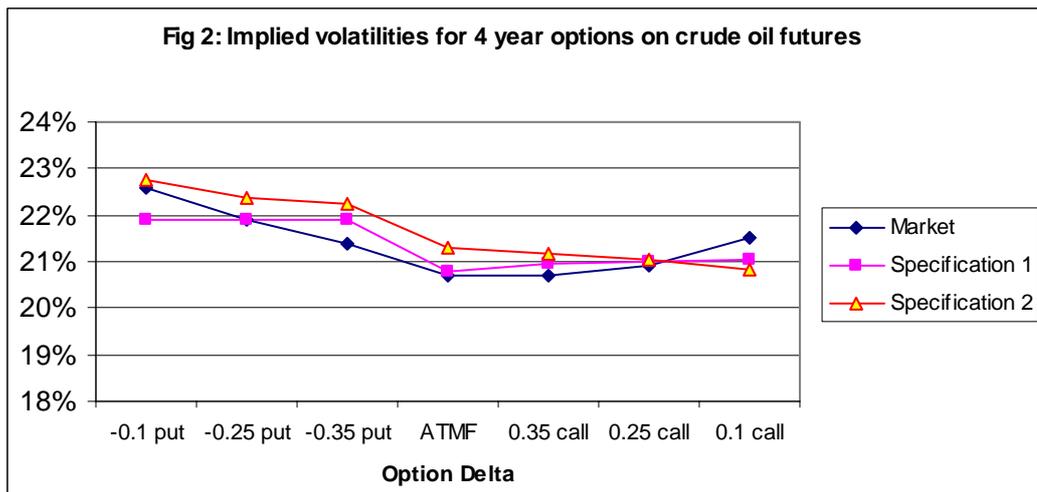
Figure 1:

Fig 1: Implied volatilities for 1 month options on crude oil futures

Figure 2:

Fig 2: Implied volatilities for 4 year options on crude oil futures

Table 1 (Example 1) No Poisson processes

$$T_2 = T_1 + 0.125$$

All options are standard European calls on futures. The values of T_1 are down the first column.
Below are the option prices

Strikes ->

	75	80	95	110	115
0.25	19.812	15.081	4.213	0.515	0.214
0.5	19.805	15.421	5.530	1.292	0.730
0.75	19.836	15.702	6.367	1.924	1.219
1	19.860	15.920	6.986	2.447	1.652
2	19.869	16.468	8.605	4.023	3.061
3	19.789	16.766	9.656	5.203	4.185

Below are the implied Black (1976) volatilities (expressed as percentages), using a price of 95.

0.25	22.525%
0.5	21.177%
0.75	20.167%
1	19.407%
2	17.789%
3	17.154%

The above option prices are in the pure-diffusion case and are priced using equation 5.12.

In all cases, $H\left(t, T_2\right) = 95$ for all T_2 and $P\left(t, T_1\right) = \exp\left(-0.05\left(T_1 - t\right)\right)$ for all T_1.

In all cases, we have two Brownian motions (in addition to the Brownian motion driving interest-rates) and

$\eta_{H1} = 0.266$, $\eta_{H2} = 0.249/1.045 \approx 0.23827751196$, $\chi_{H1} = 0.0$, $\chi_{H2} = -0.249/1.045$
$a_{H2} = 1.045$, $\sigma_r = 0.0096$, $\alpha_r = 0.2$, $\rho_{H1H2} = -0.805$, $\rho_{PH1} = -0.0964$, $\rho_{PH2} = 0.1243$

Table 2 (Example 2) One Poisson process

$$T_2 = T_1 + 0.125$$

All options are standard European calls on futures. The values of T_1 are down the first column. Below are the option prices

Strikes->	75	80	95	110	115
0.25	19.8460	15.1892	4.7491	0.9345	0.5129
0.5	19.9199	15.6447	6.0987	1.7881	1.1347
0.75	19.9956	15.9661	6.9049	2.4148	1.6419
1	20.0410	16.1943	7.4844	2.9143	2.0654
2	20.0639	16.7238	8.9826	4.3986	3.4127
3	19.9732	16.9906	9.9626	5.5164	4.4828

Below are the standard errors for the option prices above

Strikes->	75	80	95	110	115
0.25	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001
0.5	<0.0001	<0.0001	0.0001	0.0003	0.0004
0.75	0.0001	0.0002	0.0005	0.0008	0.0009
1	0.0003	0.0004	0.0009	0.0014	0.0013
2	0.0009	0.0012	0.0019	0.0025	0.0026
3	0.0011	0.0014	0.0021	0.0028	0.0028

Below are the implied Black (1976) volatilities (expressed as percentages) of the option prices above

Strikes->	75	80	95	110	115
0.25	24.107%	24.371%	25.393%	26.769%	27.258%
0.5	22.734%	22.885%	23.355%	23.882%	24.063%
0.75	21.491%	21.590%	21.874%	22.167%	22.270%
1	20.525%	20.598%	20.798%	20.988%	21.049%
2	18.442%	18.489%	18.575%	18.651%	18.676%
3	17.596%	17.633%	17.702%	17.756%	17.764%

For the above option prices, we have used equations 5.9 and 5.12 with $M = 1$ and

$$\lambda_1 = 0.75, \ \beta_1 = 0.22, \ b_1 = 2.0.$$

In all cases, $H(t, T_2) = 95$ for all T_2 and $P(t, T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

Table 3 (Example 3) Two Poisson processes

$$T_2 = T_1 + 0.125$$

All options are standard European calls on futures. The values of T_1 are down the first column.
Below are the option prices

Strikes->

	75	80	95	110	115
0.25	20.109	15.693	5.924	1.885	1.279
0.5	20.695	16.817	8.159	3.626	2.744
0.75	21.310	17.769	9.704	5.021	4.008
1	21.867	18.563	10.911	6.188	5.103
2	23.530	20.801	14.208	9.626	8.452
3	24.564	22.187	16.306	11.990	10.831

Below are the implied Black (1976) volatilities (expressed as percentages) of the option prices above

Strikes->

	75	80	95	110	115
0.25	31.022%	30.800%	31.685%	34.313%	35.195%
0.5	30.227%	30.373%	31.281%	32.531%	32.947%
0.75	29.858%	30.046%	30.785%	31.622%	31.897%
1	29.588%	29.769%	30.382%	31.020%	31.227%
2	29.015%	29.143%	29.509%	29.847%	29.953%
3	28.802%	28.900%	29.168%	29.403%	29.476%

For the above option prices, we have used equations 5.10 and 5.12 with $M = 2$ and

$$\lambda_1 = 0.75, \ \beta_1 = 0.22, \ \upsilon_1 = 0.01, \ b_1 = 0.0, \ \lambda_2 = 0.75, \ \beta_2 = -0.15, \ \upsilon_2 = 0.01, \ b_2 = 0.0$$

In all cases, $H(t,T_2) = 95$ for all T_2 and $P(t,T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

Table 4 (Example 4)

$T_1 = 2$, $P(t,T_1) \approx 0.930921801$. All options are standard European calls.
$T_2 = 2 + (13/365) \approx 2.035616438$, $H(t,T_2) = 41.02$

	Specification 1			Specification 2		
Option type	call	call	call	call	call	call
Strike	37.02	41.02	45.02	37.02	41.02	45.02
Price	7.1443	5.3267	3.9119	7.1335	5.2871	3.8473
Implied vol	24.870%	24.852%	24.828%	24.814%	24.665%	24.526%
Stan. error	0.0009	0.0008	0.0008	N/A	N/A	N/A

Table 5 (Example 5)

$T_1 = 2$, $P(t,T_1) \approx 0.930921801$. All options are standard European calls.
$T_2 = 2 + 3 + (13/365) \approx 5.035616438$, $H(t,T_2) = 28.42$

	Specification 1			Specification 2		
Option type	call	call	call	call	call	call
Strike	24.42	28.42	32.42	24.42	28.42	32.42
Price	4.6792	2.5808	1.2985	4.8958	2.7387	1.3599
Implied vol	17.123%	17.333%	17.416%	19.006%	18.400%	17.862%
Stan. error	<0.0001	<0.0001	<0.0001	N/A	N/A	N/A

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