

# Valuing inflation futures contracts

*In recent years, futures contracts written on inflation (specifically, on the ratio of the consumer price index (CPI) level at two different times) have been introduced. Working within the Jarrow & Yildirim (2003) model, John Crosby derives formulas for the theoretical values of these futures contracts in terms of the nominal and real yield curves and the CPI level*

tions guarantee the existence of a unique risk-neutral equivalent martingale measure, defined with respect to the nominal money market account numeraire, under which we denote expectations, at time  $t$ , by  $E_t[\cdot]$ . We will use  $re$  to distinguish quantities associated with real interest rates. We denote the (continuously compounded) risk-free nominal and real short rates, at time  $t$ , by  $r(t)$  and  $r_{re}(t)$  respectively. We denote nominal and real discount factors, at time  $t$  to time  $T$ , by  $P(t, T)$  and  $P_{re}(t, T)$  respectively. We denote the spot inflation index (which we will term the CPI for brevity), at time  $t$ , by  $X(t)$ .

We assume, following Jarrow & Yildirim (2003) and Mercurio (2005), that under the equivalent martingale measure, the dynamics of these quantities follow:

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt + \sigma_P(t, T)dz_P(t) \quad (1)$$

$$\begin{aligned} \frac{dP_{re}(t, T)}{P_{re}(t, T)} = & (r_{re}(t) - \rho_{re, X}\sigma_X(t)\sigma_P^{re}(t, T))dt \\ & + \sigma_P^{re}(t, T)dz_P^{re}(t) \end{aligned} \quad (2)$$

$$\frac{dX(t)}{X(t)} = (r(t) - r_{re}(t))dt + \sigma_X(t)dz_X(t) \quad (3)$$

where  $dz_P(t)$ ,  $dz_P^{re}(t)$  and  $dz_X(t)$  denote standard Brownian increments,  $\rho_{re, X}$  is the correlation (assumed constant) between  $dz_P^{re}(t)$  and  $dz_X(t)$ , and  $\sigma_X(t)$  is the volatility of the CPI index (which we assume to be deterministic). All the Brownian motions in equations (1), (2) and (3) can be correlated, although we assume these correlations are all constant. Further, we assume that the discount factor volatility terms  $\sigma_P(t, T)$  and  $\sigma_P^{re}(t, T)$  are purely deterministic functions of  $t$  and  $T$ , satisfying  $\sigma_P(T, T) \equiv 0$ . There are, potentially, different forms of  $\sigma_P(t, T)$  and  $\sigma_P^{re}(t, T)$  but we have in mind the extended Vasicek form (however, Belgrade, Benhamou & Koehler, 2004, do consider alternatives for the latter). The extended Vasicek form assumes that  $\sigma_P(t, T)$  and  $\sigma_P^{re}(t, T)$  are of the form:

$$\begin{aligned} \sigma_P(t, T) & \equiv \eta(1 - \exp(-\alpha(T-t)))/\alpha \\ \sigma_P^{re}(t, T) & \equiv \eta_{re}(1 - \exp(-\alpha_{re}(T-t)))/\alpha_{re} \end{aligned}$$

where  $\eta$ ,  $\alpha$ ,  $\eta_{re}$  and  $\alpha_{re}$  are all positive constants.

The market for inflation-linked derivatives has grown exponentially in recent years. This has resulted in futures contracts, written on inflation, being introduced in the US and Europe in order to give market participants hedging tools. These contracts also provide a vehicle for traders and speculators, such as hedge funds, to take positions that reflect their opinions about future inflation. Indeed, the most recently published data (at the time this article was originally prepared in December 2006) on US inflation (which showed an unexpectedly large increase and its biggest single monthly increase since November 1974) may serve to increase trader and investor interest in inflation futures contracts. In this article, we will derive theoretical values for these contracts. Of course, as with the three-month Libor interest rate futures contracts (which are often used to help build Libor interest rate yield curves), the primary motivation is to be able to use the market prices of these inflation futures contracts to build real interest rate yield curves. One additional benefit of our approach is that we can build real yield curves that naturally incorporate seasonality effects. We will work within the Jarrow & Yildirim (2003) model (see also Mercurio, 2005, and Belgrade, Benhamou & Koehler, 2004).

## The dynamics of nominal and real interest rates

Throughout this article, we will make the assumptions that the market is frictionless, complete and arbitrage-free. These assump-

We have written equations (1) and (2) with a single Brownian motion in each, but all results in this article also extend to multiple Brownian motions.

If we denote the forward CPI index level, at time  $t$  to time  $T$ , by  $F_X(t, T)$ , then:

$$F_X(t, T) = \frac{X(t)P_{re}(t, T)}{P(t, T)} \quad (4)$$

Further, by Itô's lemma, applied to equation (4):

$$\begin{aligned} \frac{dF_X(t, T)}{F_X(t, T)} = & \\ & \left\{ \text{cov}(\sigma_P(t, T)dz_P(t), \sigma_P(t, T)dz_P(t) \right. \\ & \left. - \sigma_X(t)dz_X(t) - \sigma_P^{re}(t, T)dz_P^{re}(t) \right\} dt \\ & + \sigma_X(t)dz_X(t) + \sigma_P^{re}(t, T)dz_P^{re}(t) - \sigma_P(t, T)dz_P(t) \end{aligned} \quad (5)$$

Notice that the drift and volatility terms in the stochastic differential equation for  $F_X(t, T)$  are deterministic and that  $F_X(t, T)$  is lognormally distributed.

#### Inflation futures

Mercurio (2005) considers the pricing of zero-coupon inflation-linked swaps and shows how they can be used to build a term structure of real discount factors in a model-independent way. One potential issue arising from this is that these swaps are quoted in the market for tenors of whole numbers of years and, in practice, the shortest maturities quoted are at least one year. This makes it difficult to build the real yield curve for tenors of less than one year or for a tenor of, say, 18 months. This is especially

pertinent as seasonality will typically have no impact on pricing inflation-linked derivatives whose maturities are whole numbers of years whereas it will affect the pricing of derivatives with a maturity of, say, six months or 18 months.

Belgrade & Benhamou (2004) consider the issue of seasonality and suggest a methodology for estimating its impact using historical data. One potential drawback to their approach is that, in order to price inflation-linked derivatives, one needs information about seasonality in the risk-neutral measure (rather than in the real-world physical measure implicit in using historical data). If we can value futures contracts on inflation, then in principle we could use futures prices to help build a term structure of real interest rates incorporating the effects of seasonality while working within the risk-neutral measure. It is the issue of valuing futures contracts on inflation that we will address in this section.

There are futures contracts on inflation in the US and in the eurozone. The US futures contracts are written on annualised three-month inflation based on the CPI-U (non-seasonally adjusted) index. This is the same index as used for US Treasury inflation-linked bonds. The eurozone futures contracts are written on the annual inflation rate in the 12-month period preceding the contract month, based on the HICPxT (non-seasonally adjusted) index.

The futures contracts are written on the ratio of the CPI index level at time  $T_2$  (which is the futures contract expiry) divided by its level at some earlier time  $T_1$ . We wish to derive a formula for the theoretical value  $H(t, T_2; T_1)$ , at time  $t$ , of such futures contracts.

We can write the delivery value  $H(T_2, T_2; T_1)$  of the futures contract at its expiry time  $T_2$  in the form (with  $T_1 < T_2$ ):

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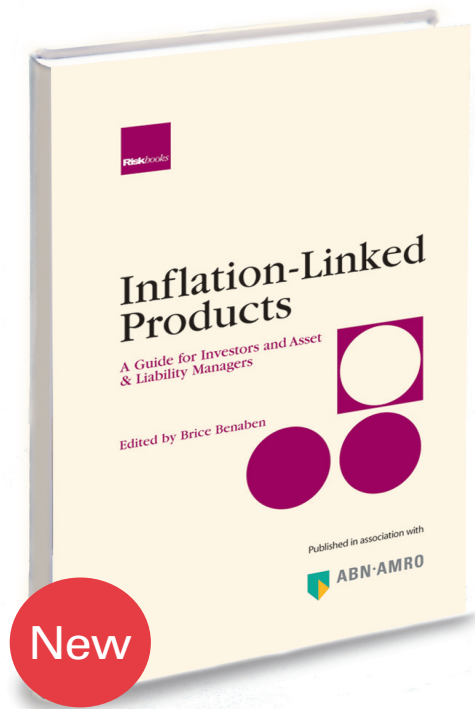
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$$H(T_2, T_2; T_1) = a - b \left( \frac{X(T_2)}{X(T_1)} - 1 \right) \quad (6)$$

The contract specifications imply that, for US CPI futures contracts, we have  $a = 100$ ,  $b = 400$  and  $T_1 = T_2 - 0.25$ , whereas for eurozone HICPxT futures contracts, we have  $a = 100$ ,  $b = 100$  and  $T_1 = T_2 - 1$  (all times are measured in years). We will now obtain an expression for the value  $H(t, T_2; T_1)$  of the futures contract, at time  $t$ . We know (by using results in Cox, Ingersoll & Ross, 1981) that, in the absence of arbitrage, the value  $H(t, T_2; T_1)$  of the futures contract, at time  $t$ , is:

$$H(t, T_2; T_1) = E_t [H(T_2, T_2; T_1)] = a + b - b E_t \left[ \frac{X(T_2)}{X(T_1)} \right] \quad (7)$$

There are two cases of interest. In the first case,  $T_1 \leq t \leq T_2$ , and in the second case,  $t < T_1 < T_2$ . We will evaluate the expectation in equation (7) in each of these two cases separately.

■ **First case.** We will consider, firstly, the case,  $T_1 \leq t \leq T_2$ . In this case, the value of  $X(T_1)$  is already known. The value of the futures contract, at time  $t$ , is:

$$\begin{aligned} H(t, T_2; T_1) &= a + b - \frac{b}{X(T_1)} E_t [X(T_2)] \\ &= a + b - \frac{b}{X(T_1)} E_t [F_X(T_2, T_2)] \end{aligned} \quad (8)$$

(by taking  $X(T_1)$  outside of the expectation and then using equation (4)). However, as noted after equation (5),  $F_X(T_2, T_2)$  is log-normally distributed. Hence, standard results can be used to evaluate the expectation and show that the value of the futures contract, at time  $t$ , for  $T_1 \leq t \leq T_2$ , is:

$$H(t, T_2; T_1) = a + b - b \frac{X(t) P_{re}(t, T_2)}{X(T_1) P(t, T_2)} \exp(\psi(t : t \geq T_1)) \quad (9)$$

where:

$$\begin{aligned} \psi(t : t \geq T_1) &\equiv \int_t^{T_2} \text{cov} \left( \sigma_P(s, T_2) dz_P(s), \right. \\ &\quad \left. \sigma_P(s, T_2) dz_P(s) - \sigma_X(s) dz_X(s) \right. \\ &\quad \left. - \sigma_P^{re}(s, T_2) dz_P^{re}(s) \right) ds \end{aligned}$$

■ **Second case.** We will consider, secondly, the case,  $t < T_1 < T_2$ . In this case, we need to calculate the value of:

$$E_t \left[ \frac{X(T_2)}{X(T_1)} \right] = E_t \left[ \frac{F_X(T_2, T_2)}{F_X(T_1, T_1)} \right]$$

(using equation (4)). But this is simply the expectation of the ratio of two lognormally distributed random variables. Hence, we can show that the value of the futures contract, at time  $t$ , for  $t < T_1 < T_2$ , is:

$$H(t, T_2; T_1) = a + b - b \frac{P_{re}(t, T_2) P(t, T_1)}{P_{re}(t, T_1) P(t, T_2)} \exp(\psi(t : t < T_1)) \quad (10)$$

where:

$$\begin{aligned} \psi(t : t < T_1) &\equiv \int_{T_1}^{T_2} \text{cov} \left( \sigma_P(s, T_2) dz_P(s), \sigma_P(s, T_2) dz_P(s) \right. \\ &\quad \left. - \sigma_X(s) dz_X(s) - \sigma_P^{re}(s, T_2) dz_P^{re}(s) \right) ds \\ &\quad - \int_t^{T_1} \text{cov} \left( \sigma_P(s, T_2) dz_P(s), \sigma_P^{re}(s, T_2) dz_P^{re}(s) - \sigma_P(s, T_2) dz_P(s) \right) ds \\ &\quad + \int_t^{T_1} \text{cov} \left( \sigma_X(s) dz_X(s), \sigma_P^{re}(s, T_1) dz_P^{re}(s) - \sigma_P^{re}(s, T_2) dz_P^{re}(s) \right) ds \\ &\quad + \int_t^{T_1} \text{cov} \left( \sigma_P^{re}(s, T_1) dz_P^{re}(s) - \sigma_P(s, T_1) dz_P(s), \sigma_P^{re}(s, T_1) dz_P^{re}(s) \right. \\ &\quad \left. - \sigma_P^{re}(s, T_2) dz_P^{re}(s) + \sigma_P(s, T_2) dz_P(s) \right) ds \end{aligned}$$

The terms  $\psi(t : t \geq T_1)$  and  $\psi(t : t < T_1)$  could loosely be termed ‘convexity adjustments’. Once a specific form has been chosen for the volatility terms appearing in equations (1) to (3) (in practice, this would most likely be by choosing the discount factor volatility terms to be of the extended Vasicek form and the CPI index volatility to be constant), then these convexity adjustments can be calculated analytically.<sup>1</sup> Examination of the forms of equations (9) and (10) suggests that, in practice, the convexity adjustments will be close to zero for inflation futures contracts of, say, less than nine months to maturity. However, the convexity adjustments will become much more significant for longer-dated<sup>2</sup> futures contracts. One would then need to calibrate the model parameters as described in Jarrow & Yildirim (2003), Mercurio (2005) and Belgrade, Benhamou & Koehler (2004).

Given the market prices of inflation futures contracts, one can then use equations (9) and (10) to build the term structure of real discount factors for those tenors corresponding to the futures contract expiry times  $T_2$  in equations (9) and (10) of successive futures contracts. For the reasons given above, this is especially relevant for capturing seasonality effects at shorter maturities. ■

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<sup>1</sup> There is no need (and not doing so saves a lot of time), when coding up these convexity adjustment formulas, to actually work out each term explicitly. For example, with discount factor volatility terms of the extended Vasicek form, the integrals can all be expressed as sums of integrals of the form:

$$\int_{t_{start}}^{t_{end}} (k_1 + C_1 \exp(-\alpha_1(T_1 - s))) (k_2 + C_2 \exp(-\alpha_2(T_2 - s))) ds$$

for constants  $k_1, k_2, C_1, C_2, \alpha_1, \alpha_2, T_1, T_2$ . This integral, which is clearly analytic, can be coded in a function that is then called repeatedly with appropriate inputs. This approach is also valid for a multi-factor version of the Jarrow & Yildirim (2003) model

<sup>2</sup> The longest-dated futures contracts on the eurozone HICPxT are currently one year although, as the market for inflation futures contracts develops, the maximum tenor might well be increased (as happened with the three month Libor interest rate futures contracts). There are currently futures contracts on US CPI up to a tenor of three years (albeit the liquidity of longer-dated contracts tends to be somewhat lower than for short-dated contracts)

## References

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