Approximating Lévy processes by a hyperexponential jump-diffusion process with a view to option pricing

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This is joint work with Nolwenn Le Saux and
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Motivation

- There is a lot of interest in pricing options using (infinite activity) Lévy processes. This is because these processes allow for volatility smiles and skews and can capture empirically observed features in time-series data such as high-frequency jumps, fat-tails, etc.

- Pricing vanilla options under Lévy processes is straightforward using Fourier inversion methods (Carr and Madan (1999), Lipton (2001), Lewis (2001)).

- Pricing exotic options is much harder. Few, if any, analytical results exist. Pricing will generally require (time-consuming) Monte Carlo simulation (which also makes estimation of partial derivatives ("Greeks") challenging).

Motivation 2

- Basic idea of this talk: Approximate the Lévy process by a hyperexponential jump-diffusion (henceforth HEJD) process (ie a jump-diffusion process with sums of double exponentially distributed compound Poisson jumps).

- Then exploit the fact that HEJD processes have a very great deal of tractability. In particular, analytical results are available (upto Laplace inversion) for first passage times, for barrier options (including double barrier options, with or without rebates), for lookback options and some other (path-dependent) exotics.

- Idea introduced by Asmussen, Madan and Pistorius (2007) and then taken further by Jeannin and Pistorius (2008). See also Carr and Crosby (2008).

- Problem: How to construct the approximating HEJD process?

- That is the question we will try to address today.

- With little loss of generality, we will only consider non-time-changed Lévy processes here.

Introduction

- Let us introduce some notation. We define the initial time (today) by t_0 and denote calendar time by t, $t \geq t_0$. Consider a market, which we assume to be free of arbitrage, where the risk-free interest rate is r and in which there is an asset, which pays a dividend yield q, whose price at time t is S_t.

- The absence of arbitrage guarantees the existence of a risk-neutral equivalent martingale measure. However, as we will utilise Lévy processes, the market is incomplete and, hence, the risk-neutral equivalent martingale measure is not unique. We will assume that one such measure $\mathbb{Q}$ has been fixed on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{Q})$. We denote by $\mathbb{E}_t^{\mathbb{Q}}[]$ the conditional expectation, under $\mathbb{Q}$, at time t.

- We assume that, under the risk-neutral measure $\mathbb{Q}$, the asset price evolves as:

$$S_t = S_{t_0} \exp((r-q)(t-t_0) + X_t),$$

where X_t is a Lévy process, mean-corrected such that $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t)] = 1$ for all $t \geq t_0$, with $X_{t_0} = 0$.

Introduction 2

- Consider a Lévy process with Lévy density $\nu(x)$ (essentially $\int_A \nu(x)dx$ tells us how many jumps, whose sizes are in the set A, occur in a unit of time). All the Lévy processes (VG, CGMY, NIG and Generalised Hyperbolic) which have been used in finance have Lévy densities which are completely monotonic (completely monotonic is a little stronger than monotonic - it means not only monotonic but also that all derivatives are monotonic and switching sign).

- We assume that the Lévy density is completely monotonic.

- Bernstein's theorem (a well-known and long-established result in mathematics) says that we can express the Lévy density in the form:

$$\nu(x) = \mathbf{1}_{\{x>0\}} \int_0^{+\infty} e^{-ux} \mu_+(u)\, du + \mathbf{1}_{\{x<0\}} \int_{-\infty}^{0} e^{-ux} \mu_-(u)\, du$$

  where $\mathbf{1}_A$ denotes the indicator function of the set A and where $\mu_+(du)$, $\mu_-(du)$ are measures on the intervals $(0, \infty)$, $(-\infty, 0)$ respectively.

Introduction 3

- Again:

$$\nu(x) = \mathbf{1}_{\{x>0\}} \int_0^{+\infty} e^{-ux} \mu_+(u) \; du + \mathbf{1}_{\{x<0\}} \int_{-\infty}^{0} e^{-ux} \mu_-(u) \; du$$

- Since the Lévy density of the jump component of a HEJD process looks like $\sum_{i=1}^{N/2} a_i b_i e^{-b_i x} \mathbf{1}_{\{x>0\}} + \sum_{i=N/2+1}^{N} c_i d_i e^{d_i x} \mathbf{1}_{\{x<0\}}$. It is immediate that we have the basis for an approximation scheme: Replace the infinite limits by large (in magnitude) finite quantities. Approximate all the corresponding very small jumps by Brownian motion (Central Limit Theorem). Approximate the integrals by discrete summations. Asmussen, Madan and Pistorius (2007) and Jeannin and Pistorius (2008) use this as the basis of their approximation scheme.

- They guessed (based on intuition) some points x at which to approximate the Lévy density $\nu(x)$. (Question: Good guesses or not?)

- They guessed (based on intuition) some mean jump sizes for the individual exponentially-distributed jumps which constituted the HEJD process. (Question: Good guesses or not?)

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- In Asmussen, Madan and Pistorius (2007), they did a non-linear least-squares fit, over the choice of the mean jump sizes and the jump intensity rates, between the Lévy density $\nu(x)$ and the Lévy density of the HEJD process evaluated at the chosen points x. (Questions: Non-linear least-squares fit stable or not? Well-posed or ill-posed?)

- In Jeannin and Pistorius (2008), they kept the mean jump sizes the same as the initial guesses and only fitted the jump intensity rates in the least-squares fit. (Questions: Good guesses or not again? Guaranteed non-negative intensity rates?)

- The diffusion component was estimated by approximating as Brownian motion all the jumps whose magnitude were less than the smallest mean jump sizes (for both up and down jumps). (Questions: What is the error here? How "small" are the "small" jumps? What happens if the Lévy process is highly skewed (for example, equity options). Intuitively and thinking of the technical conditions for the Central Limit Theorem, approximating a highly skewed process by Brownian motion will work less well than approximating a symmetric process. How "highly skewed" is "highly skewed"?)

Introduction 5

- The fitting procedures of Asmussen, Madan and Pistorius (2007) and of Jeannin and Pistorius (2008) could be described as ad-hoc or as an "engineers' solution". Its not really a criticism. After all:

- Firstly, the approximation of the Lévy density was by no means the central point of either of those papers.

- Secondly, the fitting procedures are intuitive and easy to implement.

- Thirdly, based on results reported in Jeannin and Pistorius (2008), the resulting barrier option prices are reasonably accurate.

- However, the question remains: Is there an alternative and better methodology?

What is a better methodology?

- We wish to find a more systematic methodology of approximating Lévy processes by HEJD processes which has the following six features:
- 1. No non-linear least-squares fitting is required.
- 2. No guessing of the mean jump sizes is required.
- 3. The methodology is equally as intuitive and easy to implement as the procedures described above.
- 4. The methodology has a robust way of approximating the very small jumps by Brownian motion.
- 5. The methodology yields much more accurate vanilla option prices than the procedures described above.
- 6. The methodology yields much more accurate barrier option prices than the procedures described above.

Why these criteria?

- We want to avoid non-linear least-squares fitting because often unstable and ill-posed.
- We want more accurate vanilla option prices for benchmarking purposes.
- We will focus on barrier options here. However, we believe our methodology should also be of interest for pricing other exotic options, either by Laplace transform methods (for example, lookback options) or by Monte Carlo simulation.

HEJD notation

- The hyperexponential jump-diffusion (henceforth HEJD) process, with an arbitrary number N (we assume N is even for notational simplicity) of compound Poisson processes has the form

$$X_t = \sigma W_t + \sum_{i=1}^{N/2} \sum_{k=1}^{N_{t,i}} J_{ik}^{+} - \sum_{i=N/2+1}^{N} \sum_{k=1}^{N_{t,i}} J_{ik}^{-},$$

  where W_t denotes a standard Brownian motion with $W_{t_0} = 0$ and where $N_{t,i}$, for each $i = 1, 2, ..., N$, denotes a Poisson (counting) process with $N_{t_0,i} = 0$ and random variables $J_{ik}^{+}, J_{(i+N/2)k}^{-}$, for $i = 1, ..., N/2$, $k \in \mathbb{N}$, are independent exponentially distributed.

- We denote by a_i and $c_{i+N/2}$ respectively the intensity rates of the Poisson processes corresponding to up jumps and down jumps, and we denote by b_i and $d_{i+N/2}$ the reciprocals of the mean jump sizes for the up and down jumps respectively, for each $i = 1, 2, ..., N/2$.

HEJD notation continued

- i.e.

$$\mathbb{E}^{\mathbb{Q}}_{t_0}[N_{t,i}] = a_i(t-t_0), \quad \mathbb{E}^{\mathbb{Q}}[J^+_{ik}] = \frac{1}{b_i}, \qquad \text{for } 1 \le i \le N/2,$$
$$\mathbb{E}^{\mathbb{Q}}_{t_0}[N_{t,i}] = c_i(t-t_0), \quad \mathbb{E}^{\mathbb{Q}}[J^-_{ik}] = \frac{1}{d_i}, \qquad \text{for } N/2+1 \le i \le N.$$

The characteristic exponent $\phi_N(z)$ of this process has the form (choose $\alpha = 1$ for the mean-corrected form, else $\alpha = 0$):

$$\begin{aligned}\phi_N(z) &= -\frac{1}{2}\sigma^2(z^2 + i\alpha z) \\ &+ \sum_{i=1}^{N/2}\left[a_i b_i \left(\frac{1}{b_i - iz} - \frac{1}{b_i}\right) - i\alpha z a_i b_i \left(\frac{1}{b_i - 1} - \frac{1}{b_i}\right)\right] \\ &+ \sum_{i=N/2+1}^{N}\left[c_i d_i \left(\frac{1}{d_i + iz} - \frac{1}{d_i}\right) - i\alpha z c_i d_i \left(\frac{1}{d_i + 1} - \frac{1}{d_i}\right)\right].\end{aligned}$$

Lévy process

- We consider a Lévy process $(X_t)_{t \geq t_0}$, with $X_{t_0} = 0$ a.s. Define $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(izX_t)]$ to be the characteristic function of X_t, where $z \in \mathbb{R}$. We define the characteristic exponent $\phi(z)$ via, $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(izX_t)] \equiv \exp((t - t_0)\phi(z))$, for each $t \geq t_0$.

- We define the mean-corrected characteristic exponent to be $\phi(z) - iz\phi(-i)$. It is straightforward to see that this is the characteristic exponent of a Lévy process X_t which satisfies $\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_t}\right] = 1$ (this is sufficient for applications in option pricing).

- If we only consider a Lévy process with Lévy density which decays fast enough as $|x| \rightarrow \infty$ so that the integral $\int_{\mathbb{R}\setminus[-1,1]} x\nu(x)dx$ exists (this is the case for, eg., CGMY and NIG, but not the case in general, eg., for the α-stable process) (and without Gaussian component), then from the Lévy-Khintchine formula, we can write the characteristic exponent in the form:

$$\phi(z) = \int_{-\infty}^{+\infty} (e^{izx} - 1 - i\beta zx)\nu(x)dx - i\alpha z \int_{-\infty}^{+\infty} (e^{x} - 1 - \beta x)\nu(x)dx,$$

  where $\alpha = 1$ for the mean-corrected characteristic exponent and $\alpha = 0$ otherwise and $\beta = 0$ for a process of finite variation and $\beta = 1$ otherwise.

Our aim

- Given N, we seek to approximate ϕ by the characteristic exponent ϕ_N of a HEJD process. In other words we must choose $a_i,\ b_i,\ c_i,\ d_i$ and σ^2 so that the approximation by the HEJD process is as accurate as possible.

- We do this because we know convergence in distribution follows from the convergence of corresponding characteristic exponents (or characteristic functions).

Initial presentation

- If we substitute the equation from Bernstein's theorem into the equation for the characteristic exponent and then change variables $u \to -u$ and $x \to -x$ in the integrals over $(-\infty, 0)$ and switch the order of integration, we get:

$$\phi(z) = \int_0^{+\infty} \mu_+(u) g^+_{\alpha,\beta}(u,z)\, du + \int_0^{+\infty} \mu_-(-u) g^-_{\alpha,\beta}(u,z)\, du,$$

where

$$g^{\pm}_{\alpha,\beta}(u,z) \equiv \left(\frac{1}{u \mp iz} - \frac{1}{u} \mp \frac{i\beta z}{u^2}\right) - i\alpha z \left(\frac{1}{u \mp 1} - \frac{1}{u} \mp \frac{\beta}{u^2}\right).$$

- We recognize that the terms $g^+_{\alpha,\beta}(u,z)$ and $g^-_{\alpha,\beta}(u,z)$ are of the same form as the summands appearing in the characteristic exponent of a HEJD process, where the exponentially distributed jumps have mean sizes $1/u$ and the compound Poisson processes have intensity rates equal to $1/u$.

- The terms $-i\beta z/u^2$ and $i\beta z/u^2$ are simply additional (and, from an option pricing perspective, irrelevant) drift terms, which in any event cancel out if $\alpha = 1$ or $\beta = 0$.

Initial presentation 2

- We define $h^{\pm}_{\alpha,\beta}(u,z) \equiv -g^{\pm}_{\alpha,\beta}(u,z)/(z^2+i\alpha z)$, introduce $\overline{\theta}_+,\ \overline{\theta}_- \in \mathbb{R}_+$ and express the exponent ϕ as

$$\begin{aligned}\phi(z) &= \int_0^{\overline{\theta}_+} \mu_+(u) g^+_{\alpha,\beta}(u,z)\, du + \int_0^{\overline{\theta}_-} \mu_-(-u) g^-_{\alpha,\beta}(u,z)\, du \\ &- (z^2+i\alpha z)[\int_{\overline{\theta}_+}^{+\infty} \mu_+(u) h^+_{\alpha,\beta}(u,z)\, du + \int_{\overline{\theta}_-}^{+\infty} \mu_-(-u) h^-_{\alpha,\beta}(u,z)\, du].\end{aligned}$$

- This suggests an approximation scheme where the first two integrals are replaced by sums:

$$\begin{aligned}\phi(z) &\simeq \sum_{i=1}^{N/2} \omega_i^+ \mu_+(u_i^+) g^+_{\alpha,\beta}(u_i^+,z) + \sum_{i=1+\frac{N}{2}}^{N} \omega_i^- \mu_-(-u_i^-) g^-_{\alpha,\beta}(u_i^-,z) \\ &- \frac{1}{2}(\Sigma^+ + \Sigma^-)(z^2+i\alpha z).\end{aligned}$$

Initial presentation 3

- Here ω_i^+ and u_i^+, for $i = 1, ..., N/2$, are respectively weights and abscissas coming from a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_+)$, ω_i^- and u_i^-, for $i = 1 + N/2, ..., N$, are respectively weights and abscissas coming from a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_-)$, and:

$$\Sigma^+ \equiv \int_{\overline{\theta}_+}^{+\infty} \mu_+(u) h_{\alpha,\beta}^+(u, z) du, \quad \Sigma^- \equiv \int_{\overline{\theta}_-}^{+\infty} \mu_-(-u) h_{\alpha,\beta}^-(u, z) du.$$

- Observing the form of the last equation we see that we have written the characteristic exponent of the Lévy process in a form that resembles the characteristic exponent of a HEJD process. In terms of the parameters a_i, b_i, c_i and d_i, we have:

$$\begin{aligned} a_i b_i &= \omega_i^+ \mu_+(u_i^+) \quad \text{and} \quad b_i = u_i^+, \ 1 \le i \le N/2 \\ c_i d_i &= \omega_i^- \mu_-(-u_i^-) \quad \text{and} \quad d_i = u_i^-, \ 1 + N/2 \le i \le N. \end{aligned}$$

- However, there are two sources of error in our proposed approximation.

Error terms

- 1. The discretization error: We replaced the integrals by finite sums.

- 2. The truncation error: We would like the integrands $\mu_+(u)h^+_{\alpha,\beta}(u,z)$ and $\mu_-(-u)h^-_{\alpha,\beta}(u,z)$ and hence Σ^+ and Σ^- to be independent of z. However, clearly, neither Σ^+ nor Σ^- is independent of z and this prevents us from identifying the parameter σ^2 as being equal to $\Sigma^+ + \Sigma^-$. Note that for a Lévy density ν it can be shown easily that $\mu_+(u)$ and $\mu_-(-u)$ must grow slower than a quadratic in u, as $u \to \infty$, and therefore, observing the forms of $h^+_{\alpha,\beta}(u,z)$ and $h^-_{\alpha,\beta}(u,z)$, we have $\lim_{\overline{\theta}_\pm \to \infty} \Sigma^\pm = 0$. Hence the error in the third and fourth terms could be viewed as a truncation error in the upper limit of the integrals.

- Note how these two errors work in opposite directions.

Truncation error

- Let us now have a closer look at the term $(\Sigma^+ + \Sigma^-)\,(z^2 + i\alpha z)/2$. We momentarily view this term as the characteristic exponent of a random variable and then calculate the second and third central moments μ_2 and μ_3 of this random variable by differentiating 2 or 3 times. We obtain:

$$\mu_2 \equiv 2\int_{\overline{\theta}_+}^{+\infty} \frac{\mu_+(u)}{u^3} du + 2\int_{\overline{\theta}_-}^{+\infty} \frac{\mu_-(-u)}{u^3} du,$$

$$\mu_3 \equiv 6\int_{\overline{\theta}_+}^{+\infty} \frac{\mu_+(u)}{u^4} du - 6\int_{\overline{\theta}_-}^{+\infty} \frac{\mu_-(-u)}{u^4} du.$$

- Its easy to see that if $\overline{\theta}_+$ and $\overline{\theta}_-$ are large enough, then $|\mu_3| \ll \mu_2$. Furthermore, in the mean-corrected case $\alpha = 1$, a simple calculation shows that the term $(\Sigma^+ + \Sigma^-)\,(z^2 + i\alpha z)/2$ behaves asymptotically like

$$(z^2 + iz)\left[\int_{\overline{\theta}_+}^{\infty} \mu_+(u)\left(\frac{1}{u^3}\right)\, du + \int_{\overline{\theta}_-}^{\infty} \mu_-(-u)\left(\frac{1}{u^3}\right)\, du\right] = \frac{1}{2}\mu_2(z^2 + iz)$$

if $\overline{\theta}_+$ and $\overline{\theta}_-$ are both much, much larger than $|z|$. We recognize the right-hand side as the mean-corrected characteristic exponent of Brownian motion with variance μ_2.

Truncation error 2

- Our approximation, therefore, is to replace the term $(\Sigma^+ + \Sigma^-)(z^2 + i\alpha z)/2$ in our equation for the characteristic exponent by the term $\mu_2(z^2 + i\alpha z)/2$. This is intuitively equivalent to approximating small jumps (both up and down) by Brownian motion with variance μ_2.

- In order for us to justify this approximation, we would want the term $\mu_2(z^2 + i\alpha z)/2$ to be as close as possible to $(\Sigma^+ + \Sigma^-)(z^2 + i\alpha z)/2$. When $z = 0$, both terms equal zero and hence there is no approximation. This suggests that, in order to get a handle on the truncation error, we need to compare $(\Sigma^+ + \Sigma^-)(z^2 + i\alpha z)/2$ and $\mu_2(z^2 + i\alpha z)/2$ when evaluated at some value of z, z_{large} say, such that $|z|$ is large. Once we have chosen z_{large}, this suggest a measure of the truncation error TE:

$$TE \equiv \frac{1}{2}|\mu_2(z_{\text{large}}^2 + i\alpha z_{\text{large}}) - \left(\Sigma^+ + \Sigma^-\right)(z_{\text{large}}^2 + i\alpha z_{\text{large}})|.$$

Discretisation error

- We estimate the integrals $\int_0^{\overline{\theta}_+} \mu_+(u) g^+_{\alpha,\beta}(u,z) du$ and $\int_0^{\overline{\theta}_-} \mu_-(-u) g^-_{\alpha,\beta}(u,z)\, du$ by a numerical method such as a Gauss-Legendre quadrature rule, using some number of points N_{large} where $N_{\text{large}} \gg N$. We then take the (modulus of the) difference between these (very accurate) estimates and those obtained by a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_+)$ and by a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_-)$.

- We have to choose a value of z at which the integrals are computed. From the definitions of $g^+_{\alpha,\beta}(u,z)$ and $g^-_{\alpha,\beta}(u,z)$, we know that when $z = 0$, $g^+_{\alpha,\beta}(u,0)$ and $g^-_{\alpha,\beta}(u,0)$ are both identically equal to zero for all u. Hence, in order to get a meaningful estimate of the discretization error, we need to evaluate the integrands at some value of z such that $|z|$ is large. We elect to evaluate them at the same z_{large} that we use in estimating the truncation error. Hence, we get an estimate for the discretization error.

Initial estimates

- Now we need a way to choose the limits $\overline{\theta}_+$ and $\overline{\theta}_-$ of the integrals. Since, intutively speaking, the errors act in opposite directions, a possible criterion is to find $\overline{\theta}_+$ and $\overline{\theta}_-$ such that the discretization error DE and the truncation error TE are equal, using, for example, a solver-type methodology. In other words, we search for $\overline{\theta}_+$ and $\overline{\theta}_-$ such that $|TE - DE|^2$ is minimised and we do so in the hope that this minimum is zero.

- Why is the criterion of trying to minimise (or hopefully uniquely set to zero) $|TE - DE|^2$ a sensible one?

Initial estimates 2

- Once we find $\overline{\theta}_+$ and $\overline{\theta}_-$ such that the discretization and truncation errors are equal, we can get ω_i^+ and u_i^+, for $i = 1, ..., N/2$, from a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_+)$ and likewise we can get ω_i^- and u_i^-, for $i = 1 + N/2, ..., N$, from a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_-)$. We can then immediately get estimates for a_i, b_i, for $1 \leq i \leq N/2$, c_i, d_i, for $1 + N/2 \leq i \leq N$

$$\begin{aligned} a_i b_i &= \omega_i^+ \mu_+(u_i^+) \quad \text{and} \quad b_i = u_i^+,\ 1 \leq i \leq N/2 \\ c_i d_i &= \omega_i^- \mu_-(-u_i^-) \quad \text{and} \quad d_i = u_i^-,\ 1 + N/2 \leq i \leq N. \end{aligned}$$

and for σ^2, via $\sigma^2 = \mu_2$.

Simplification

- The strategy just described is certainly feasible and it would be applicable to any Lévy process (with a completely monotonic Lévy density) but it does rely on being able to solve uniquely for the limits $\overline{\theta}_+$ and $\overline{\theta}_-$ via a solver-type methodology.

- It would be preferable to simplify the algorithm in order to reduce the problem to solving for a single parameter using a simple one-dimensional root finder method such as bisection.

- To do this, we must make a further assumption about the Lévy density (for this and the next slide only), namely that it can be expressed in the form:

$$\begin{aligned} L(x)\exp(-r_+x) &= \int_0^\infty e^{-ux}\mu_+(u)du \quad \text{for } x > 0, \\ L(-x)\exp(r_-x) &= \int_{-\infty}^0 e^{-ux}\mu_-(u)du \quad \text{for } x < 0, \end{aligned}$$

  where $L : (0, \infty) \to \mathbb{R}$ is completely monotonic and $r_+, r_- \in \mathbb{R}$ are constants. In words, we assume that the Lévy density is symmetric apart from a (possibly asymmetric) exponential tilting.

- CGMY, NIG, VG and Generalised Hyperbolic processes all satisfy this assumption.

Simplification 2

- With this simplifying assumption (and omitting a few details for brevity), it is clear that we can exploit the natural asymmetry. We make an obvious change of variables $u \longrightarrow u - r_+$ or $u \longrightarrow u - r_-$ in the integrals. Then, we assume that $\overline{\theta}_+ - r_+ = \overline{\theta}_- - r_- \equiv \overline{\theta}$, say.

- Now we equate the discretization error and the truncation error and we only have to solve for a single parameter $\overline{\theta}$.

- Hence, we need only a simple bisection method to solve for $\overline{\theta}$.

- We can show that the equation obtained from setting the discretization error and truncation error equal certainly has a single unique root.

Summary so far

- Whether using the simplified algorithm (bisection) or the two-dimensional algorithm (solver), we now have estimates for a_i, b_i, $c_{i+N/2}$, $d_{i+N/2}$, $i = 1, \ldots, N/2$ and σ^2 which are essentially analytic. It is shown in Le Saux (2008) (Nolwenn's dissertation) that these estimates would allow us to compute prices of vanilla options under a HEJD process which are quite close to the prices of vanilla options under the Lévy process in equation.

- However, the prices are not as close as we would like. Therefore, we now seek to refine our parameter estimates.

- A key point is to refine only the estimates for the parameters that enter linearly into the characteristic exponent (i.e. a_i, $c_{i+N/2}$, and σ^2). Therefore from now on we regard the mean jump sizes $1/b_i$ and $1/d_{i+N/2}$ as fixed.

- For each $i = 1, \ldots, N/2$ we denote by $a_i^{(0)}$, $c_{i+N/2}^{(0)}$ the initial estimates of a_i, $c_{i+N/2}$ and by $\sigma^{(0)2}$ the initial value σ^2.

Refining the results

- We now seek to refine our initial estimates by finding the values of a_i, $c_{i+N/2}$, for $i = 1, \ldots, N/2$, and σ^2 which most closely match (in both real and imaginary parts) the characteristic exponent of a HEJD process (multiplied by a carefully chosen weighting function $z \mapsto \Omega(z)$) with the characteristic exponent of the Lévy process in question (multiplied by the same weighting function $z \mapsto \Omega(z)$) at some judiciously chosen points z_k, $k = 1, ..., m$, in $\mathbb{C}$.

- Essentially, we now have to solve a linear system of the form $Ax = b$, where $A \in \mathbb{R}^{2m\times(N+1)}$, $x \in \mathbb{R}^{N+1}$ and $b \in \mathbb{R}^{2m}$, where $x = [a_1 \ldots a_{N/2}\ c_{1+N/2} \ldots c_N\ \sigma^2]^T$, and where, for $1 \leq k \leq m$, b_{2k-1} and b_{2k} are given by the real and imaginary parts of $\Omega(z_k)\phi(z_k)$ respectively.

- Further, for $1 \leq k \leq m$, $1 \leq j \leq N+1$, $A_{2k-1,j}$ and $A_{2k,j}$ are respectively given by the real and imaginary parts of:

$$\begin{cases} \Omega(z_k)\left(\left(\dfrac{b_j}{b_j - iz_k} - 1\right) - i\alpha z_k\left(\dfrac{b_j}{b_j - 1} - 1\right)\right) & \text{if } 1 \leq j \leq \frac{N}{2}, \\ \Omega(z_k)\left(\left(\dfrac{d_j}{d_j + iz_k} - 1\right) - i\alpha z_k\left(\dfrac{d_j}{d_j + 1} - 1\right)\right) & \text{if } \frac{N}{2} + 1 \leq j \leq N, \\ -\Omega(z_k)\left(\dfrac{z_k^2 + i\alpha z_k}{2}\right) & \text{if } j = N+1. \end{cases}$$

Tikhonov regularisation

- If we try to solve this linear system directly, we will have two problems.

- Firstly, since we fit both the real and imaginary part of the characteristic exponents, we will have an even number of equations $2m$. If we decide to fit $a_i, c_{i+N/2}$, $i = 1, \ldots, N/2$, and σ^2 the number of parameters will be odd, and the linear system will not be square. In any event, we may wish to have the flexibility to choose m such that $2m > N + 1$.

- Secondly, solving the linear system directly does not guarantee that $a_i, c_{i+N/2}$ and σ^2 are all positive.

- To allow us to fit the characteristic exponents at a number of points m, possibly such that $2m > N + 1$, and to **try to ensure** a_i, $c_{i+N/2}$ and σ^2 are all positive, we will use Tikhonov regularization. We seek x that minimizes $|Ax - b|^2 + \varepsilon^2 |x - x_0|^2$, where $\varepsilon \in \mathbb{R}^+$ and where $x_0 \in \mathbb{R}^{N+1}$ is the vector of our initial estimates $x_0 = [a_1^{(0)} \ldots a_{N/2}^{(0)} \; c_{1+N/2}^{(0)} \ldots c_N^{(0)} \; \kappa\sigma^{(0)2}]^T$. The solution is given by:

$$x = x_0 + (A^T A + \varepsilon^2 I)^{-1} A^T (b - Ax_0).$$

Vanilla options

- We still need to choose the the points z_k, for $k = 1, ..., m$, where we try to match the characteristic exponents, as well as z_{large} and the form of the weighting function $\Omega(z)$.

- Intutition: Use the use the form of the Lipton (2001) Fourier inversion pricing formula for vanilla options to make these choices (full details in the paper).

Summary so far

- The criterion of setting the discretisation error equal to the truncation error gives us, essentially, the value at which we truncate the small jumps of the Lévy process and replace them by Brownian motion. Once we have this value, we get analytical estimates for the intensity rates, mean jump sizes and Brownian component volatility (which turn out to be very good estimates).

- We then refine the estimates for the intensity rates and volatility by matching (i.e. attempting to equate) the characteristic exponent of the Lévy process and that of the approximating HEJD process. This is also essentially analytic (involves only inverting a matrix).

- How does it fare in practice?

Vanilla comparisons

- We used the NIG parameters (Eurostoxx 50 equity index) from Jeannin and Pistorius (2008).

- We valued vanilla options with an initial asset price $S_{t_0} = 100$ and time to maturity equal to one year. We priced options with 41 different strikes where the strikes were of the form $100\exp(y)$ where the value of y ranged from -0.8 to 0.8 in intervals of 0.04. Hence, the strikes varied from approximately 44.93 to approximately 222.55. For the options with strikes greater than or equal to 100, we valued call options, else we valued put options. We then converted these prices to implied volatilities (expressed as percentages).

- We price vanilla option prices under the NIG process using five different approaches:

- Approach (a): Using the vanilla option pricing formula (Lipton (2001)) with the characteristic function for the NIG process. Clearly this approach will give us benchmark values.

- For the remaining four approaches, we used the vanilla option pricing formula with the characteristic function for the HEJD process where we have fitted $N = 14$ Poisson processes (seven up and seven down).

Vanilla comparisons 2

- Approach (b): We used a solver-type methodology to find the roots $\overline{\theta}_+$ and $\overline{\theta}_-$ of the equation obtained by setting the truncation errors and discretisation errors equal. We then refined our estimates by attempting to match the characteristic exponents.

- Approach (c): We proceeded as in approach (b). We then further revised our estimate for σ^2 by exactly matching the variance of the NIG process and of the HEJD process. The estimates for a_i, b_i, $c_{i+N/2}$, $d_{i+N/2}$, $i = 1, \dots, N/2$, are exactly as in approach (b).

- Approach (d): We used the intensity rates, mean jump sizes and diffusion volatility from Jeannin and Pistorius (2008).

- Approach (e): We used the simplified algorithm where we only do a one-dimensional search for $\overline{\theta}$ followed by refining our estimates by attempting to match the characteristic exponents.

Vanilla comparisons 3

- Approach (e) (simplified algorithm) works the best.
- Approach (d) (Jeannin and Pistorius (2008)) works least well.
- Moment matching (approach (c)) does not seem to help.

Barrier option comparisons

- Our approach is as follows. We approximate the Lévy process by a HEJD process. We then price barrier options using Carr and Crosby (2008) together with Laplace inversion. We will refer to this methodology as the HEJDCC methodology.

- Boyarchenko and Levendorskii (2008a) and (2008b) (henceforth the BoyarLeven methodology) have developed FFT-based algorithms for pricing barrier options under General Classes of Lévy processes which are not based on approximating the Lévy process by a HEJD process. The BoyarLeven methodology uses numerical methods and hence can't be and won't be literally exact. However, it does not approximate the Lévy process in question by a HEJD process at the outset as our approach does and it does appear to give accurate barrier option prices. Hence, we will use prices to benchmark the accuracy of our HEJDCC methodology.

- When available, we will also compare prices against Jeannin and Pistorius (2008) (we will refer to their methodology as the JPHEJD methodology).

Single barrier option comparisons under NIG

- We price down-and-out put (single) barrier options with the barrier set at 2100. The options have a strike of 3500 and a maturity of one year. We price the options with 32 different initial asset prices which are expressed as a percentage of 3500. The percentages are: 64.0, 66.0,...,126.0. Hence, the initial asset prices varied from 2240 to 4410.

- The NIG parameters are the same as we used for vanillas and we used approaches (b) and (e) again.

- Again, fitted a HEJD process with fourteen Poisson processes (seven up and seven down).

Single barrier option comparisons under NIG 2

- Almost prefect agreement between HEJDCC and BoyarLeven.

- The root-mean-square proportional errors in the HEJDCC methodology are about one-fourteenth the root-mean-square errors in the JPHEJD methodology.

- We believe that the reason for the better performance of the HEJDCC methodology is that the procedure for fitting a HEJD process to the NIG process is much better.

Double barrier options, alternatives processes such as CGMY

- In the paper, we show that the methodology works equally well:
- For CGMY (not just NIG).
- For both finite variation (for CGMY, $Y < 1$) and infinite variation processes (for CGMY, $1 \leq Y < 2$).
- And for double barrier options of different types.

Conclusions

- We have shown that, by approximating the Lévy process in question by a HEJD process, we can very accurately price barrier options (as long as the initial asset price is not very close to the barrier (or barriers)).

- We have illustrated that our methodology for approximating the Lévy process by a HEJD process yields more accurate barrier option prices than the methodology of Jeannin and Pistorius (2008).

- Although we have considered barrier options, we believe our methodology will be relevant for pricing other types of options both by Laplace transform methods and by Monte Carlo simulation.

- The paper that we have written (and Nolwenn's dissertation) can be found on my website: http://www.john-crosby.co.uk . (Also the paper is on Aleksandar's website).