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Commodities: A simple Multi-factor Jump-Diffusion Model

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Abstract

In this paper, we develop an arbitrage-free model for the pricing and risk management of commodity derivatives. The model generates futures (or forward) commodity prices consistent with any initial term structure. The model is consistent with mean reversion in commodity prices, which is an empirically observed stylised fact about commodity markets, and it also generates stochastic convenience yields. Our model is a multi-factor jump-diffusion model, one version of which allows for long-dated futures contracts to jump by smaller amounts than short-dated futures contracts, which is in line with stylised empirical observations. Finally, our model also allows for stochastic interest-rates. The model produces semi-analytic solutions for standard European options. This opens the possibility to calibrate the model parameters by deriving implied parameters from the market prices of options.

1. Introduction

The aim of this paper is to develop an arbitrage-free multi-factor jump-diffusion model for commodities. The commodity could be, for example, crude oil, another petroleum product, gold, a base metal, natural gas or electricity.

Before turning our attention to commodities, it is worth reflecting on the development of interest-rate models. The paper by Vasicek (1977) introduced an equilibrium mean reverting interest-rate model into the literature. By introducing a time-dependent mean reversion level, this became the extended Vasicek model (Babbs (1990), Hull and White (1993)) which could automatically fit any initial term structure of interest-rates. Black et al. (1990) used a similar idea in a non-Gaussian setting. These models focused principally on instantaneous short rates. Further research (Babbs (1990), Heath et al. (1992)) developed no-arbitrage models (including multi-factor versions) evolving the entire yield curve, consistent with its initial values. There are many parallels between the above interest-rate models and modelling futures (or forward) commodity prices. Some of the commodities literature (Gibson and Schwartz (1990)) has focussed on equilibrium models with the first factor being the spot commodity price and the second factor being the instantaneous convenience yield whilst Schwartz (1997) introduced a third factor, with stochastic interest-rates. However these models leave the market price of convenience yield risk to be determined in equilibrium and are not necessarily consistent with any initial term structure of futures (or forward) prices. Subsequent models (by analogous techniques to interest-rate modelling) have been consistent with any initial term structure. See for example, Cortazar and Schwartz (1994), Carr and Jarrow (1995), Beaglehole and Chebanier (2002), Miltersen and Schwartz (1998), Miltersen (2003), Clewlow and Strickland (2000),(1999) with the latter particularly focussing on evolving the forward price curve. In this respect, our paper is closest in spirit to Clewlow and Strickland (1999) though we also incorporate stochastic interest-rates (and jumps).

As a general rule, attention has mostly focused on pure diffusion models. Jumps were incorporated into interest-rate models in Babbs and Webber (1994),(1997), Bjork et al. (1997) and Jarrow and Madan (1995). See also Merton (1976),(1990), Hoogland et al. (2001), Duffie et al. (2000) and Runggaldier (2002). Paralleling these models, jumps have also been introduced into models for commodity prices in Hilliard and Reis (1998), Deng (1998), Clewlow and Strickland (2000), Benth et al. (2003) and Casassus (2004) although usually in models which are not necessarily consistent with any initial term structure.

Our model will attempt to introduce a multi-factor jump-diffusion model, with stochastic interest-rates, which is consistent with any initial term structure. We now turn our attention to this model by, firstly, outlining features of the commodities markets.

It is an empirical fact (Bessembinder et al. (1995)) that most commodity prices seem to exhibit mean reversion. Furthermore, it is also empirically observed, in the case of electricity, that the prices of short-dated (close to delivery) contracts exhibit sharp spikes. The impact of these price spikes is much lower for contracts with a greater time to delivery. Other commodities, such as oil and natural gas, can also exhibit price spikes although these tend to be of a smaller magnitude. However, we also observe that the market prices of options on many commodities imply Black and Scholes (1973)/Black (1976) volatilities which vary with the strike of the option. That is, market prices imply a volatility smile or (more usually) a volatility skew. One way to account for volatility smiles and skews is through a jump-diffusion model.

Unlike financials assets, which are held for investment purposes, commodities are held in order to be consumed or used in an industrial process (although gold and, to some extent, other precious metals can be and are held for investment purposes, they are also used for some specialist industrial purposes). The notion of convenience yield is introduced for commodities. Loosely speaking, it is a measure of the value of physically holding a commodity rather than being long the commodity through the forward or futures markets. For example, an end-user of a commodity may well choose to store some of it (as a type of self-insurance policy) in order to minimise disruption if there is a problem with supply. The convenience yield also implicitly accounts for the cost of storage of the commodity and the cost of insuring the commodity. It is observed empirically that convenience yields are usually highly volatile. Furthermore, convenience yields are usually positively correlated with the value of the commodity (Lence and Hayes (2002)).

There is a macro-economic interpretation to mean reversion and convenience yields through linkage to supply and demand and inventory levels:
When prices are low, some producers may stop producing, which will tend to cause prices to rise. If prices are high, some consumers may stop consuming which will tend to cause prices to fall. When inventories are low, shortages are more likely, which tends to increase both the value of the commodity and the perceived value of physically holding the commodity (as opposed to being long the commodity through the forward or futures markets). This latter can be interpreted as increasing the convenience yield. The reverse argument holds if inventories are high.

We would like our model to incorporate all of the above stylised observations of the commodities markets. Examination of our multi-factor jump-diffusion model will show that it captures all of the above effects.

We will also assume that interest-rates are stochastic. When interest-rates are stochastic, futures commodity prices and forward commodity prices are no longer the same. In this paper, we will work with both futures and forward prices but mostly with futures commodity prices.

We will assume that markets are frictionless. That is, continuous trading is possible and we assume that there are no bid-offer spreads in the commodities markets or in the bond markets. Of course, we do not assume that the commodity can be stored or insured without cost since it is precisely these costs which give rise to the notion of convenience yield.

We will assume that markets are free of arbitrage.

It is well known (Harrison and Pliska (1981), Duffie (1996)) that, under these assumptions, there exists an equivalent martingale measure under which futures prices are martingales. In the case of a diffusion model, if there are sufficient futures contracts (and risk-free bonds and, possibly, forward contracts) traded, then any derivative (such as an option) can be instantaneously hedged or replicated by a dynamic self-financing portfolio of futures contracts (and risk-free bonds). The market in our model is thus complete. In this case, the equivalent martingale measure is unique. However, in the case of a jump-diffusion model, the market may be either complete or incomplete. If the market is incomplete then the equivalent martingale measure would not be unique. In the case of incompleteness, we will assume that an equivalent martingale measure is "fixed by the market" through the market prices of options and we will call this (by an abuse of language but for the sake of brevity) the equivalent martingale measure (rather than an equivalent martingale measure). It is also possible for the jump-diffusion model to lead to a market which is complete. The circumstances in which the jump-diffusion model gives rise to a complete market are specified in section 6.

The remainder of this paper is structured as follows. In section 2, we will provide notation and introduce the model. In section 3, we will relate it to stochastic convenience yields and to mean reverting commodity prices. In section 4, we will discuss how the model can be used in connection with Monte Carlo simulation. In section 5, we will derive the prices of standard options, in semi-analytical form, in our model. In section 6 (which, at least partially, logically precedes section 2), we derive no-arbitrage conditions for our model, explain the circumstances under which our model leads to complete and incomplete markets, derive partial integro-differential equations satisfied by the price of commodity derivatives and relate futures commodity prices to forward commodity prices. Section 7 is a short conclusion.

2. The model of futures commodity prices

Notation:

Let us explain some notation. All jump-diffusion processes are assumed right continuous. More explicitly, $H(t,T) = \lim_{u \downarrow t} H(u,T)$ includes the effect of any jump at time t. The value of $H(t,T)$ just before a jump at time t is $H(t-,T) = \lim_{u \uparrow t} H(u,T)$. When we write $\frac{dH(t,T)}{H(t,T)}$ in a SDE, we mean $\frac{dH(t,T)}{H(t-,T)}$. For the sake of brevity however, we shall always write $\frac{dH(t,T)}{H(t,T)}$.

We define today to be time t_0 and we denote calendar time by t, ($t \geq t_0$).
In this section and in sections 3 to 5, we will work exclusively in the equivalent martingale measure (which as already indicated may, in fact, not be unique). As already indicated, if the equivalent martingale measure is not unique, we will assume that one has been "fixed by the market" and we will call this the (rather than a) equivalent martingale measure.

We denote expectations, at time t, with respect to the equivalent martingale measure by $Exp_t[\ \ ]$.

Stochastic evolution of interest-rates:

We assume that interest-rates in our model are stochastic. Let us introduce some notation.

We denote the (continuously compounded) risk-free short rate, at time t, by $r(t)$, we denote the (continuously compounded) instantaneous forward rate, at time t, to time T by $f(t,T)$ and we denote the price, at time t, of a (credit risk free) zero coupon bond maturing at time T by $P(t,T)$.

By definition $P(t,T) = \exp\left(-\int_t^T f(t,s)ds \right)$.

All references to bond prices in this and subsequent sections are, of course, references to risk-free zero coupon bond prices.

We assume that (under the equivalent martingale measure) the short rate follows the extended Vasicek process, (Babbs (1990), Hull and White (1990),(1993)) namely,

$$dr(t) = \alpha_r(\gamma(t) - r(t))dt - \sigma_r dz_P(t),$$

or equivalently (Babbs (1990), Heath et al. (1992)) the dynamics of bond prices are

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \frac{\sigma_r}{\alpha_r}(1 - \exp(-\alpha_r(T-t)))dz_P(t) \equiv r(t)dt + \sigma_P(t,T)dz_P(t).$$

(equation 2.1)

Note that $dz_P(t)$ denotes standard Brownian increments. We assume that σ_r and α_r are positive constants and $\gamma(t)$ is defined so as to be consistent with the initial term structure (ie the term structure of interest rates today, time t_0), which we take as given.

Define the state variable $X_P(t) = \int_{t_0}^{t} \sigma_r \exp(-\alpha_r(t-s))dz_P(s)$, (note $X_P(t_0) = 0$).

(equation 2.2)

It can be shown (Babbs (1990)) that:

$$r(t) \equiv f(t,t) = f(t_0,t) + \int_{t_0}^{t} \sigma_P(s,t)\frac{\partial \sigma_P(s,t)}{\partial t}ds - X_P(t) \qquad \text{(equation 2.3)}$$

Commodities:

We denote the value of the commodity at time t by C_t. The value of the commodity today is C_{t_0}. The value of the commodity is usually termed the spot price. However, in this paper, we shall generally use the expression "value of the commodity" because, in some commodity markets, the spot price is not always exactly easy to define.

Now we turn our attention to futures commodity prices.

We denote the forward commodity price, at time t, to (ie for delivery at) time T, by $F(t,T)$.

We denote the futures commodity price, at time t, to (ie the futures contract matures at) time T, by $H(t,T)$.

It can be shown (Cox et al. (1981), Duffie (1996)), that in the absence of arbitrage, that

$$F(t,T) = \frac{Exp_t\left[\exp\left(-\int_t^T r(s)ds\right)C_T\right]}{P(t,T)} \qquad \text{(equation 2.4)}$$

and

$$H(t,T) = Exp_t[C_T] \qquad \text{(equation 2.5)}$$

where, to repeat, C_T is the value of the commodity at time T.

A key to modelling commodity prices when interest-rates are stochastic is to recognise that, in this case, futures commodity prices and forward commodity prices are not the same. Indeed equations 2.4 and 2.5 show that futures prices are martingales with respect to the equivalent martingale measure whereas, when interest-rates are stochastic, forward prices are not.

Note that equations 2.4 and 2.5 are consistent with

$$F(t,t) = C_t = H(t,t) \text{ and } F(T,T) = C_T = H(T,T) \qquad \text{(equation 2.6)}$$

We take as given our initial term structure (ie the term structure today, time t_0) of futures commodity prices. That is, we know $H(t_0,T)$ for all T of interest, ($T \geq t_0$) (perhaps, in practice, by interpolation of the futures prices of a finite number of futures contracts).

In some models, the dynamics of the value of the commodity are posited and then equations 2.4 and 2.5 would be used to derive the dynamics of forward commodity prices and futures commodity prices. By contrast, our model will posit the dynamics of futures commodity prices. In other words, futures contracts are not derivatives but, instead, are the primitive assets of our model.

We will shortly posit the dynamics of futures commodity prices $H(t,T)$ in the equivalent martingale measure (consistent with the martingale property of equation 2.5). We will then obtain the dynamics of the value of the commodity via the relation $C_t = H(t,t)$.

Let us consider why equation 2.6 has to be valid:
Consider a futures contract[1] maturing at time T.
If it were the case that $C_T < H(T,T)$, then it would be possible to create a risk-less arbitrage by buying the commodity and selling the futures contract. Now, whilst it is easily possible to make short sales of financial assets, short sales of commodities are either very difficult or (more likely) impossible. Therefore we assume that there are non-satiated agents in the commodities markets (for example, oil companies, mining companies, oil refineries, industrial end-users of commodities, etc) who hold the commodity in strictly positive quantities and who would sell their holdings if it was profitable to do so. With this assumption, if it were the case that $C_T > H(T,T)$, then these agents would create a risk-less arbitrage by selling their holdings of the commodity and buying the futures contract. Since our model assumes no arbitrage, it must be the case that $C_T = H(T,T)$. Likewise $C_t = H(t,t)$.

Now we introduce the instantaneous futures convenience yield forward rate $\varepsilon(t,T)$, at time t to time T via the relation

$$H(t,T) = \frac{C_t}{P(t,T)} \exp\left(- \int_{s=t}^{T} \varepsilon(t,s) ds \right). \qquad \text{(equation 2.7)}$$

Define $P_\varepsilon(t,T) = \exp\left(- \int_{s=t}^{T} \varepsilon(t,s) ds \right)$. (equation 2.8)

[1] We note that we ignore any impact of "quality", "timing" and "location" options which are sometimes embedded in commodities futures contracts.

This defines what we call a fictitious futures convenience yield bond price. We call it fictitious because no such bond actually exists, nor do we assume it exists. It is solely a mathematical construction defined, by analogy to interest-rates and real risk-free bonds, via equations 2.7 and 2.8.

We can also write $H(t,T) = \dfrac{C_t P_\varepsilon(t,T)}{P(t,T)}$. (equation 2.9)

We introduce K standard Brownian increments denoted by $dz_{Hk}(t)$, for each k, $k = 1,2,...,K$. We denote the correlation between $dz_P(t)$ and $dz_{Hk}(t)$ by ρ_{PHk}, for each k, and the correlation between $dz_{Hk}(t)$ and $dz_{Hj}(t)$ by ρ_{HkHj} for each j and k, $j,k = 1,2,...,K$, and $\rho_{HkHj} = 1$ if $k = j$.

We also introduce M Poisson processes denoted by N_{mt}, for each m, $m = 1,...,M$, with $N_{mt_0} \equiv 0$, whose intensity rates are $\lambda_m(t)$. We assume that $\lambda_m(t)$ are deterministic functions of at most t and they must be positive, for each m, $m = 1,...,M$, for all t. We also assume that each of the N_{mt} are independent of each other and each is independent of each of the Brownian motions.

We introduce $b_m(t)$, for each m, $m = 1,...,M$, which are non-negative deterministic functions. We call these jump decay coefficient functions.

We introduce γ_{mt}, for each m, $m = 1,...,M$, which are parameters which determine the size of the jump, conditional on a jump in N_{mt}. We will call the γ_{mt} the spot jump amplitudes.
At risk of complication, but for the sake of brevity, we will consider two possible specifications for the spot jump amplitudes, and in turn, these are linked to two possible specifications of the jump decay coefficient functions.

For each m, $m = 1,...,M$, we assume that either:

Assumption 2.1 :
The spot jump amplitudes are assumed to be (known) constants. In this case, the jump decay coefficient functions $b_m(t)$ are assumed to be any non-negative deterministic functions. •

Or:

Assumption 2.2 :
The spot jump amplitudes are assumed to be independent and identically distributed random variables, each of which is independent of each of the Brownian motions and of each of the Poisson processes. In this case, the jump decay coefficient functions $b_m(t)$ are assumed to be identically equal to zero ie $b_m(t) \equiv 0$ for all t. •

Remark 2.3 : Note that for each m, we assume either assumption 2.1 or assumption 2.2 is satisfied. For different m it could be a different assumption (ie if we have more than one Poisson process, we can mix the assumptions).

Remark 2.4 : The motivation for these assumptions will be described in depth in section 6 but we can provide a brief summary here. We will show that it is not possible in general, in the absence of arbitrage, to have both jumps whose amplitudes are random variables and simultaneously have jump decay coefficient functions ($b_m(t)$) which are not identically zero. Hence we assume that all the Poisson processes satisfy either assumption 2.1 or assumption 2.2. We will develop the model with assumptions 2.1 and 2.2 in parallel since the choice of these assumptions scarcely alters the development.

For each m, E_{Nmt} denotes the expectation operator, at time t, conditional on a jump occurring in N_{mt}. If, for a given m, the spot jump amplitude is constant (assumption 2.1), the expectation operator is set equal to its argument.

We are motivated by the presence of $P(t,T)$ in the denominator of equation 2.9, the effect of applying Ito's lemma to $\dfrac{C_t P_\varepsilon(t,T)}{P(t,T)}$ and by the knowledge that futures commodity prices are martingales in the equivalent martingale measure.

Assumption 2.5 :
We assume that the dynamics of futures prices in the equivalent martingale measure are:

$$\frac{dH(t,T)}{H(t,T)} = \sum_{k=1}^{K} \sigma_{Hk}(t,T)dz_{Hk}(t) - \sigma_P(t,T)dz_P(t)$$
$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)du\right)\right)-1\right)dN_{mt}$$
$$-\sum_{m=1}^{M}\lambda_m(t)E_{Nmt}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)du\right)\right)-1\right)dt \qquad \text{(equation 2.10)}$$

where $\sigma_{Hk}(t,T)$, for each k, $k=1,2,...,K$, are deterministic functions of t and T and are independent of $H(t,T)$. •

Remark 2.6 : Futures commodity prices are martingales in the equivalent martingale measure.

Remark 2.7 : In the absence of jumps, the dynamics of futures commodity prices in the equivalent martingale measure are very similar to those of forward prices in Clewlow and Strickland (1999) (although we also incorporate stochastic interest-rates). When $K=2$ (and in the absence of jumps), equation 2.10 gives dynamics for futures commodity prices which are essentially identical to those in Miltersen and Schwartz (1998) although they make the starting point of their model, the dynamics of spot commodity prices and convenience yields.

Although we index the spot jump amplitudes γ_{mt} with t, the assumptions 2.1 and 2.2 both imply that their outcomes do not depend on t, ie the index simply refers to the time at which a jump may occur.

Note that $\lambda_m(t)$ is the risk-neutral (ie under the equivalent martingale measure) intensity rate of N_{mt}, for each m and furthermore (in the case of assumption 2.2) the distributions of the spot jump amplitudes γ_{mt} are also defined with respect to the risk-neutral equivalent martingale measure. We assume that at any given instant no more than one of the M Poisson processes jumps.

Define, for each m,

$$e_m(t,T) \equiv \lambda_m(t) E_{Nmt}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du\right)\right)-1\right) \qquad \text{(equation 2.11)}$$

Note this expression is deterministic, irregardless of whether the spot jump amplitudes are as in assumption 2.1 or in assumption 2.2.

By the form of Ito's lemma for jump-diffusions, applied to equation 2.10, and using equation 2.11,

$$\begin{aligned} d(\ln H(t,T)) &= -\frac{1}{2}\left\{\sum_{k=1}^{K}\sigma_{Hk}^2(t,T)+\sigma_P^2(t,T)-2\sum_{k=1}^{K}\rho_{PHk}\sigma_P(t,T)\sigma_{Hk}(t,T)\right\}dt \\ &-\frac{1}{2}\left\{\sum_{k=1}^{K}\sum_{j=1}^{k-1}2\rho_{HkHj}\sigma_{Hk}(t,T)\sigma_{Hj}(t,T)\right\}dt+\sum_{k=1}^{K}\sigma_{Hk}(t,T)dz_{Hk}(t)-\sigma_P(t,T)dz_P(t) \\ &+\sum_{m=1}^{M}\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du\right)dN_{mt}-\sum_{m=1}^{M}e_m(t,T)\,dt \end{aligned} \qquad \text{(equation 2.12)}$$

and where we have used the usual convention that if the upper index is less than the lower index in a summation, then the sum is set to zero.

Remark 2.8 : Equation 2.12 enables us to better describe the size of the jump when one happens. When there is a jump in N_{mt}, $\ln H(t,T)$ changes by $\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du\right)$. Let us briefly consider the implications of this. When there is a jump, the log of the futures commodity prices infinitesimally close to maturity jump by γ_{mt}. However, the log of the futures commodity prices for delivery $(T-t)$ years ahead jump by $\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du\right)$. Considering the limit, as $(T-t)\to\infty$, (and provided $\exp\left(-\int_t^T b_m(u)\,du\right)\to 0$), then very long-dated futures commodity prices do not jump at all. The effect of the function $b_m(t)$, (which is assumed always non-negative), is to exponentially dampen the effect of the jump through futures commodity price tenor. This seems to be in line with empirical observations in the commodities markets (this is particularly a feature in the case of electricity). In the case of assumption 2.2, $b_m(t)\equiv 0$ and jumps cause parallel shifts in the log of the futures commodity prices across different tenors.

Remark 2.9 : Note that, for each m, γ_{mt} can take any value in $(-\infty,\infty)$ and the futures commodity price $H(t,T)$ will remain positive.

Let us return to the model:

Now rewrite equation 2.12, our SDE for $\ln H(t,T)$, for $\ln H(s,t)$ instead, and then rewrite in integral form from t_0 to t, then:

$$\ln H(t,t) = \ln H(t_0,t) - \int_{t_0}^{t}\frac{1}{2}\left\{\sum_{k=1}^{K}\sigma_{Hk}^2(s,t)+\sigma_P^2(s,t)-2\sum_{k=1}^{K}\rho_{PHk}\sigma_P(s,t)\sigma_{Hk}(s,t)\right\}ds$$

$$-\int_{t_0}^{t}\frac{1}{2}\left\{\sum_{k=1}^{K}\sum_{j=1}^{k-1}2\rho_{HkHj}\sigma_{Hk}(s,t)\sigma_{Hj}(s,t)\right\}ds+\int_{t_0}^{t}\sum_{k=1}^{K}\sigma_{Hk}(s,t)\,dz_{Hk}(s)-\int_{t_0}^{t}\sigma_P(s,t)\,dz_P(s)$$

$$+\int_{t_0}^{t}\sum_{m=1}^{M}\gamma_{ms}\exp\left(-\int_{s}^{t}b_m(u)\,du\right)dN_{ms}-\int_{t_0}^{t}\sum_{m=1}^{M}e_m(s,t)\,ds \qquad \text{(equation 2.13)}$$

By differentiating with respect to t, we get the dynamics of the value of the commodity, $C_t \equiv \ln H(t,t)$:

$$d(\ln H(t,t))=\frac{\partial \ln H(t_0,t)}{\partial t}dt+\left(\int_{t_0}^{t}-\frac{1}{2}\left\{\sum_{k=1}^{K}\left(2\sigma_{Hk}(s,t)\frac{\partial\sigma_{Hk}(s,t)}{\partial t}\right)+2\sigma_P(s,t)\frac{\partial\sigma_P(s,t)}{\partial t}\right\}ds\right)dt$$

$$+\left(\int_{t_0}^{t}-\frac{1}{2}\left\{\sum_{k=1}^{K}\sum_{j=1}^{k-1}2\rho_{HkHj}\left(\sigma_{Hk}(s,t)\frac{\partial\sigma_{Hj}(s,t)}{\partial t}+\sigma_{Hj}(s,t)\frac{\partial\sigma_{Hk}(s,t)}{\partial t}\right)\right\}ds\right)dt$$

$$+\left(\int_{t_0}^{t}\frac{1}{2}\left\{\sum_{k=1}^{K}2\rho_{PHk}\sigma_P(s,t)\frac{\partial\sigma_{Hk}(s,t)}{\partial t}\right\}ds\right)dt+\left(\int_{t_0}^{t}\frac{1}{2}\left\{\sum_{k=1}^{K}2\rho_{PHk}\sigma_{Hk}(s,t)\frac{\partial\sigma_P(s,t)}{\partial t}\right\}ds\right)dt$$

$$+\left(\int_{t_0}^{t}\sum_{k=1}^{K}\frac{\partial\sigma_{Hk}(s,t)}{\partial t}dz_{Hk}(s)-\int_{t_0}^{t}\frac{\partial\sigma_P(s,t)}{\partial t}dz_P(s)\right)dt$$

$$-\frac{1}{2}\left\{\sum_{k=1}^{K}\sigma_{Hk}^2(t,t)+\sigma_P^2(t,t)-2\sum_{k=1}^{K}\rho_{PHk}\sigma_P(t,t)\sigma_{Hk}(t,t)+\sum_{k=1}^{K}\sum_{j=1}^{k-1}2\rho_{HkHj}\sigma_{Hk}(t,t)\sigma_{Hj}(t,t)\right\}dt$$

$$+\sum_{k=1}^{K}\sigma_{Hk}(t,t)dz_{Hk}(t)-\sigma_P(t,t)dz_P(t)$$

$$-\left(\int_{t_0}^{t}\sum_{m=1}^{M}\gamma_{ms}b_m(t)\exp\left(-\int_{s}^{t}b_m(u)\,du\right)dN_{ms}\right)dt+\sum_{m=1}^{M}\gamma_{mt}dN_{mt}$$

$$-\left(\sum_{m=1}^{M}e_m(t,t)\right)dt-\left(\int_{t_0}^{t}\sum_{m=1}^{M}\frac{\partial e_m(s,t)}{\partial t}ds\right)dt \qquad \text{(equation 2.14)}$$

Note that the final diffusion term vanishes, ie $\sigma_P(t,t)dz_P(t)=0$, since $\sigma_P(t,t)=0$.

Note that in general, (examining the fourth line of equation 2.14), $\ln H(t,t)$ would be non-Markovian but we would like

$$\int_{t_0}^{t}\sum_{k=1}^{K}\frac{\partial\sigma_{Hk}(s,t)}{\partial t}dz_{Hk}(s)-\int_{t_0}^{t}\frac{\partial\sigma_P(s,t)}{\partial t}dz_P(s)$$

to be such that $\ln H(t,t)\equiv\ln C_t$ is a Markov process in a finite number of state variables.

We consider the functional form

$$\sigma_{Hk}(s,t)=\eta_{Hk}(s)+\chi_{Hk}(s)\exp\left(-\int_{s}^{t}a_{Hk}(u)du\right), \qquad \text{(equation 2.15)}$$

for each k, $k = 1,2,...,K$, where $\eta_{Hk}(s)$, $\chi_{Hk}(s)$ and $a_{Hk}(u)$ are deterministic functions[2].

Recall (equation 2.2) the state variable $X_P(t) = \int_{t_0}^{t} \sigma_r \exp(-\alpha_r (t-s)) dz_P(s)$.

And define the state variables:

$$Y_P(t) = \int_{t_0}^{t} \sigma_r dz_P(s), \tag{equation 2.16}$$

$$X_{Hk}(t) = \int_{t_0}^{t} \chi_{Hk}(s) \exp\left(-\int_{s}^{t} a_{Hk}(u)\, du \right) dz_{Hk}(s), \tag{equation 2.17}$$

$$Y_{Hk}(t) = \int_{t_0}^{t} \eta_{Hk}(s) dz_{Hk}(s). \tag{equation 2.18}$$

Note $Y_P(t_0) = 0$, and $X_{Hk}(t_0) = 0$, $Y_{Hk}(t_0) = 0$, for all k.

Define, for each m, $m = 1,...,M$,

$$X_{Nm}(t) = \int_{t_0}^{t} \gamma_{ms} \exp\left(-\int_{s}^{t} b_m(u)\, du \right) dN_{ms}, \text{ (note } X_{Nm}(t_0) = 0\text{).} \tag{equation 2.19}$$

Then $dX_{Nm}(t) = -\left(b_m(t) \int_{t_0}^{t} \gamma_{ms} \exp\left(-\int_{s}^{t} b_m(u)\, du \right) dN_{ms} \right) dt + \gamma_{mt} dN_{mt}$.

(equation 2.20)

Then we can show (since $\sigma_P(t,t) = 0$ and using equations 2.2 and 2.3) that

$$\begin{aligned}
&\left(\int_{t_0}^{t} \sum_{k=1}^{K} \frac{\partial \sigma_{Hk}(s,t)}{\partial t} dz_{Hk}(s) - \int_{t_0}^{t} \frac{\partial \sigma_P(s,t)}{\partial t} dz_P(s) \right) dt + \sum_{k=1}^{K} \sigma_{Hk}(t,t) dz_{Hk}(t) - \sigma_P(t,t) dz_P(t) \\
&= \sum_{k=1}^{K} dX_{Hk}(t) + \sum_{k=1}^{K} dY_{Hk}(t) - X_P(t) dt \\
&= \sum_{k=1}^{K} dX_{Hk}(t) + \sum_{k=1}^{K} dY_{Hk}(t) + \left(r(t) - f(t_0,t) - \int_{t_0}^{t} \sigma_P(s,t) \frac{\partial \sigma_P(s,t)}{\partial t} ds \right) dt
\end{aligned}$$

(equation 2.21)

Then using equation 2.13 and equations 2.2, 2.16, 2.17, 2.18 and 2.19, we have the following expression for the value of the commodity C_t at time t, $C_t \equiv H(t,t)$:

[2] We note that it will become clear later that in order to avoid a potential degeneracy we may put $\eta_{Hk}(t) \equiv 0$ for all k except one, (or combine terms of the form $\eta_{Hk}(t)\, dz_{Hk}(t)$) but we will write out equations below in full to ease notation.

$$H(t,t)=H(t_0,t)\exp\left(\int_{t_0}^{t}-\frac{1}{2}\left\{\sum_{k=1}^{K}\sigma_{Hk}^2(s,t)+\sigma_P^2(s,t)-2\sum_{k=1}^{K}\rho_{PHk}\sigma_P(s,t)\sigma_{Hk}(s,t)\right\}ds\right)$$
$$\exp\left(\int_{t_0}^{t}-\frac{1}{2}\left\{\sum_{k=1}^{K}\sum_{j=1}^{k-1}2\rho_{HkHj}\sigma_{Hk}(s,t)\sigma_{Hj}(s,t)\right\}ds\right)\exp\left(\sum_{k=1}^{K}\left(X_{Hk}(t)+Y_{Hk}(t)\right)+\frac{1}{\alpha_r}\left(X_P(t)-Y_P(t)\right)\right)$$
$$\exp\left(\sum_{m=1}^{M}X_{Nm}(t)-\sum_{m=1}^{M}\int_{t_0}^{t}e_m(s,t)\,ds\right) \qquad \text{(equation 2.22)}$$

In a similar manner, we can obtain the following expression for the evolution, from time t_0 to time t, of the futures commodity price to time T, in terms of the state variables:

$$H(t,T)=H(t_0,T)\exp\left(\int_{t_0}^{t}-\frac{1}{2}\left\{\sum_{k=1}^{K}\sigma_{Hk}^2(s,T)+\sigma_P^2(s,T)-2\sum_{k=1}^{K}\rho_{PHk}\sigma_P(s,T)\sigma_{Hk}(s,T)\right\}ds\right)$$
$$\exp\left(\int_{t_0}^{t}-\frac{1}{2}\left\{\sum_{k=1}^{K}\sum_{j=1}^{k-1}2\rho_{HkHj}\sigma_{Hk}(s,T)\sigma_{Hj}(s,T)\right\}ds\right)$$
$$\exp\left(\sum_{k=1}^{K}Y_{Hk}(t)+\sum_{k=1}^{K}\left[\exp\left(-\int_{t}^{T}a_{Hk}(u)du\right)X_{Hk}(t)\right]\right)$$
$$\exp\left(\frac{\exp(-\alpha_r(T-t))}{\alpha_r}X_P(t)-\frac{1}{\alpha_r}Y_P(t)\right)$$
$$\exp\left(\sum_{m=1}^{M}\left(\exp\left(-\int_{t}^{T}b_m(u)\,du\right)X_{Nm}(t)\right)-\sum_{m=1}^{M}\int_{t_0}^{t}e_m(s,T)\,ds\right) \qquad \text{(equation 2.23)}$$

This shows that $H(t,t)\equiv C_t$ and $H(t,T)$ are Markov in a finite number of state variables[3].

Remark 2.10 : With the help of results in section 4 (specifically equation 4.6), it is straightforward to verify by direct calculation using equations 2.22 and 2.23 that

$Exp_t\left[C_T\right]\equiv Exp_t\left[H(T,T)\right]=H(t,T)$ which confirms consistency with equation 2.5.

[3]We note that it is straightforward to combine the $Y_{Hk}(t)$ and $Y_P(t)$ into a single state variable. We could do this, but prefer not to, in order to maximise the intuition behind the model. However, it shows that $H(t,t)\equiv C_t$ and $H(t,T)$ are, in fact, Markovian in $K+2+M$ state variables.

3. Stochastic convenience yields and mean reverting commodity prices

Our aim in this section is to give results about stochastic convenience yields and mean reversion in our model which show that our model is able to capture the stylised observations of the commodities markets that were made in section 1.

Firstly, we provide a mathematical lemma.

Lemma 3.1 : $\dfrac{\partial \ln H(t_0,t)}{\partial t} = f(t_0,t) - \varepsilon(t_0,t).$ (equation 3.1)

Proof : We note, from equation 2.9, that

$$H(t_0,t) = \frac{C_{t_0}}{P(t_0,t)} \exp\left(-\int_{s=t_0}^{t} \varepsilon(t_0,s)\,ds \right) = C_{t_0} \exp\left(\int_{s=t_0}^{t} \left(f(t_0,s) - \varepsilon(t_0,s) \right) ds \right)$$

Now take logs and then the partial derivative with respect to t. •

Proposition 3.2 : The dynamics of the value of the commodity are as follows. If we define

$$\begin{aligned}
\varepsilon_r(t) &\equiv \varepsilon(t_0,t) - \left(\int_{t_0}^{t} -\frac{1}{2}\left\{ \sum_{k=1}^{K} \left(2\sigma_{Hk}(s,t)\frac{\partial \sigma_{Hk}(s,t)}{\partial t} \right) + 4\sigma_P(s,t)\frac{\partial \sigma_P(s,t)}{\partial t} \right\} ds \right) \\
&- \left(\int_{t_0}^{t} -\frac{1}{2}\left\{ \sum_{k=1}^{K}\sum_{j=1}^{k-1} 2\rho_{HkHj} \left(\sigma_{Hk}(s,t)\frac{\partial \sigma_{Hj}(s,t)}{\partial t} + \sigma_{Hj}(s,t)\frac{\partial \sigma_{Hk}(s,t)}{\partial t} \right) \right\} ds \right) \\
&- \left(\int_{t_0}^{t} \frac{1}{2}\left\{ \sum_{k=1}^{K} 2\rho_{PHk}\sigma_P(s,t)\frac{\partial \sigma_{Hk}(s,t)}{\partial t} \right\} ds \right) - \left(\int_{t_0}^{t} \frac{1}{2}\left\{ \sum_{k=1}^{K} 2\rho_{PHk}\sigma_{Hk}(s,t)\frac{\partial \sigma_P(s,t)}{\partial t} \right\} ds \right) \\
&- \left(\int_{t_0}^{t} \sum_{k=1}^{K} \frac{\partial \sigma_{Hk}(s,t)}{\partial t} dz_{Hk}(s) \right) \\
&+ \left(\int_{t_0}^{t} \sum_{m=1}^{M} \gamma_{ms} b_m(t) \exp\left(-\int_{s}^{t} b_m(u)\,du \right) dN_{ms} \right) + \left(\int_{t_0}^{t} \sum_{m=1}^{M} \frac{\partial e_m(s,t)}{\partial t} ds \right)
\end{aligned}$$ (equation 3.2)

then

$$\begin{aligned}
d\left(\ln H(t,t)\right) &= \left(r(t) - \varepsilon_r(t) \right) dt \\
&- \frac{1}{2}\left\{ \sum_{k=1}^{K} \sigma_{Hk}^2(t,t) + \sum_{k=1}^{K}\sum_{j=1}^{k-1} 2\rho_{HkHj}\sigma_{Hk}(t,t)\sigma_{Hj}(t,t) \right\} dt \\
&+ \sum_{k=1}^{K} \sigma_{Hk}(t,t)\,dz_{Hk}(t) + \sum_{m=1}^{M} \gamma_{mt}\,dN_{mt} - \left(\sum_{m=1}^{M} e_m(t,t) \right) dt\,.
\end{aligned}$$ (equation 3.3)

and

$$\frac{dC_t}{C_t} = \left(r(t) - \varepsilon_r(t) \right) dt + \sum_{k=1}^{K} \sigma_{Hk}(t,t)\,dz_{Hk}(t)$$

$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\right)-1\right)dN_{mt}-\left(\sum_{m=1}^{M}e_m\left(t,t\right)\right)dt. \qquad \text{(equation 3.4)}$$

Proof : Put equation 3.2 into equation 2.14, then with some algebra and equations 2.21 and 3.1, we obtain equation 3.3. Using Ito's lemma for jump-diffusions gives equation 3.4 •

Remark 3.3 : Note that the SDE in equation 3.4 has a drift term which (by construction) is of an entirely familiar form.

In order to get a greater intuition to the model, we are also interested in the dynamics of the fictitious futures convenience yield bond price $P_{\varepsilon}\left(t,T\right)$ which we display in proposition 3.4 .

Proposition 3.4 : The dynamics of the fictitious futures convenience yield bond price are:

$$\frac{dP_{\varepsilon}\left(t,T\right)}{P_{\varepsilon}\left(t,T\right)}=\mu_{\varepsilon}\left(t\right)dt-\sum_{k=1}^{K}\chi_{Hk}\left(t\right)\left(1-\exp\left(-\int_t^T a_{Hk}\left(u\right)du\right)\right)dz_{Hk}\left(t\right)$$

$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\left(\exp\left(-\int_t^T b_m\left(u\right)du\right)-1\right)\right)-1\right)dN_{mt}-\sum_{m=1}^{M}\left(e_m\left(t,T\right)-e_m\left(t,t\right)\right)dt,$$

(equation 3.5)

where

$$\mu_{\varepsilon}\left(t\right)\equiv\varepsilon_r\left(t\right)+\sum_{k=1}^{K}\left(\eta_{Hk}\left(t\right)+\chi_{Hk}\left(t\right)\right)^2+\sum_{k=1}^{K}\sum_{j=1}^{k-1}\left(2\rho_{HkHj}\left(\eta_{Hk}\left(t\right)+\chi_{Hk}\left(t\right)\right)\left(\eta_{Hj}\left(t\right)+\chi_{Hj}\left(t\right)\right)\right)$$

$$+\sum_{k=1}^{K}\left(\rho_{PHk}\left(\eta_{Hk}(t)+\chi_{Hk}(t)\exp\left(-\int_t^T a_{Hk}(u)du\right)\right)\sigma_P(t,T)\right)-\sigma_P^2(t,T)$$

$$-\sum_{k=1}^{K}\left(\left(\eta_{Hk}(t)+\chi_{Hk}(t)\exp\left(-\int_t^T a_{Hk}(u)du\right)\right)\left(\eta_{Hk}(t)+\chi_{Hk}(t)\right)\right)$$

$$-\sum_{k=1}^{K}\sum_{j=1}^{k-1}\left(2\rho_{HkHj}\left(\eta_{Hk}\left(t\right)+\chi_{Hk}\left(t\right)\exp\left(-\int_t^T a_{Hk}\left(u\right)du\right)\right)\left(\eta_{Hj}\left(t\right)+\chi_{Hj}\left(t\right)\right)\right)$$

Proof : From equation 2.9, we have $P_{\varepsilon}\left(t,T\right)=\dfrac{H\left(t,T\right)P\left(t,T\right)}{C_t}$. Now use Ito's lemma for jump-diffusions. •

Remark 3.5 : Whilst this expression appears quite long, it is conceptually straightforward as the volatility term for the Brownian motions in the SDE has a similar form to that in the SDE for risk-free bond prices in a K factor Gaussian interest-rate model (Babbs (1990), Heath et al. (1992)). Of course, the drift of a risk-free bond (or any non-dividend paying traded asset), in the equivalent martingale measure, is equal to the risk-free short rate. This does not apply to the drift of the fictitious futures convenience yield bond price however since it is a mathematical construction, not the price of a real traded asset. In addition, we see that fictitious futures convenience yield bond prices exhibit jumps, except in the special case that for all m the jump decay coefficient functions $b_m\left(t\right)$ are identically equally to zero for all t.

Proposition 3.6 : The dynamics of the instantaneous futures convenience yield forward rate $\varepsilon(t,T)$, at time t to time T , are:

$$d\varepsilon(t,T)=\left(-\sum_{k=1}^{K}\rho_{PHk}\sigma_P(t,T)\frac{\partial\sigma_{Hk}(t,T)}{\partial T}-\sum_{k=1}^{K}\rho_{PHk}\sigma_{Hk}(t,T)\frac{\partial\sigma_P(t,T)}{\partial T}\right)dt$$
$$+\left(2\sigma_P(t,T)\frac{\partial\sigma_P(t,T)}{\partial T}\right)dt$$
$$+\left(\sum_{k=1}^{K}\sigma_{Hk}(t,T)\frac{\partial\sigma_{Hk}(t,T)}{\partial T}+\sum_{k=1}^{K}\sum_{j=1}^{k-1}\rho_{HkHj}\left(\sigma_{Hk}(t,T)\frac{\partial\sigma_{Hj}(t,T)}{\partial T}+\sigma_{Hj}(t,T)\frac{\partial\sigma_{Hk}(t,T)}{\partial T}\right)\right)dt$$
$$+\sum_{k=1}^{K}\chi_{Hk}(t)a_{Hk}(T)\left(\exp\left(-\int_t^T a_{Hk}(u)du\right)\right)dz_{Hk}(t)$$
$$+\sum_{m=1}^{M}\left(\gamma_{mt}b_m(T)\exp\left(-\int_t^T b_m(u)\,du\right)dN_{mt}\right)+\sum_{m=1}^{M}\frac{\partial e_m(t,T)}{\partial T}dt \qquad \text{(equation 3.6)}$$

Proof : Apply Ito's lemma to equation 3.5 with (from equation 2.8) $\varepsilon(t,T)=-\dfrac{\partial\ln P_\varepsilon(t,T)}{\partial T}$. •

Proposition 3.7 : The dynamics of the futures convenience yield short rate $\varepsilon(t,t)$ (using terminology analogous to interest-rates) are:

$$\varepsilon(t,t)\equiv\varepsilon(t_0,t)-\left(\int_{t_0}^{t}-\frac{1}{2}\left\{\sum_{k=1}^{K}\left(2\sigma_{Hk}(s,t)\frac{\partial\sigma_{Hk}(s,t)}{\partial t}\right)+4\sigma_P(s,t)\frac{\partial\sigma_P(s,t)}{\partial t}\right\}ds\right)$$
$$-\left(\int_{t_0}^{t}-\frac{1}{2}\left\{\sum_{k=1}^{K}\sum_{j=1}^{k-1}2\rho_{HkHj}\left(\sigma_{Hk}(s,t)\frac{\partial\sigma_{Hj}(s,t)}{\partial t}+\sigma_{Hj}(s,t)\frac{\partial\sigma_{Hk}(s,t)}{\partial t}\right)\right\}ds\right)$$
$$-\left(\int_{t_0}^{t}\frac{1}{2}\left\{\sum_{k=1}^{K}2\rho_{PHk}\sigma_P(s,t)\frac{\partial\sigma_{Hk}(s,t)}{\partial t}\right\}ds\right)-\left(\int_{t_0}^{t}\frac{1}{2}\left\{\sum_{k=1}^{K}2\rho_{PHk}\sigma_{Hk}(s,t)\frac{\partial\sigma_P(s,t)}{\partial t}\right\}ds\right)$$
$$-\left(\int_{t_0}^{t}\sum_{k=1}^{K}\frac{\partial\sigma_{Hk}(s,t)}{\partial t}dz_{Hk}(s)\right)$$
$$+\left(\int_{t_0}^{t}\sum_{m=1}^{M}\gamma_{ms}b_m(t)\exp\left(-\int_s^t b_m(u)\,du\right)dN_{ms}\right)+\left(\int_{t_0}^{t}\sum_{m=1}^{M}\frac{\partial e_m(s,t)}{\partial t}ds\right).$$

Proof : We note that $\chi_{Hk}(t)a_{Hk}(T)\left(\exp\left(-\int_t^T a_{Hk}(u)du\right)\right)=-\dfrac{\partial\sigma_{Hk}(t,T)}{\partial T}$, and then rewrite our SDE for $\varepsilon(t,T)$ for $\varepsilon(s,t)$ instead, and then re-arrange terms, and then rewrite this SDE in integral form from t_0 to t. •

Remark 3.8 : We note this expression for $\varepsilon(t,t)$ is the same as the expression given earlier for $\varepsilon_r(t)$ in equation 3.2 (which indeed it should be), ie $\varepsilon_r(t) \equiv \varepsilon(t,t)$. This justifies our notation for $\varepsilon_r(t)$ and $\varepsilon(t,T)$ (ie it justifies our choice of $\varepsilon_r(t)$ in equation 3.2 and shows its consistency with equation 2.7). Note that the futures convenience yield short rate $\varepsilon(t,t)$, at time t, follows a mean reverting jump-diffusion process driven by K Brownian motions and M Poisson processes.

In section 1, we noted that empirical evidence supports the view that the value of a commodity is positively correlated with convenience yields. The following proposition derives this correlation.

Proposition 3.9 : The correlation at time t between the log of the value of the commodity and the instantaneous futures convenience yield forward rate $\varepsilon(t,T)$ is given by

$$C\,\mathrm{ov}\left(d\varepsilon(t,T), d\left(lnC_t\right)\right) \Big/ \sqrt{\left(V\,\mathrm{ar}\left(d\varepsilon(t,T)\right)\right)\left(V\,\mathrm{ar}\left(d\left(\ln C_t\right)\right)\right)},$$

where

$$C\,\mathrm{ov}\left(d\varepsilon(t,T), d\left(\ln C_t\right)\right)$$
$$= \left(\sum_{k=1}^{K}\sum_{j=1}^{K} \rho_{HkHj}\chi_{Hk}(t)\, a_{Hk}(T)\exp\left(-\int_t^T a_{Hk}(u)\,du\right)\left(\eta_{Hj}(t)+\chi_{Hj}(t)\right)\right)dt$$
$$+\sum_{m=1}^{M} Var\left(\gamma_{mt} b_m(T)\exp\left(-\int_t^T b_m(u)\,du\right)dN_{mt}\gamma_{mt}dN_{mt}\right)$$

and

$$V\,\mathrm{ar}\left(d\varepsilon(t,T)\right) = \sum_{k=1}^{K}\sum_{j=1}^{K} \rho_{HkHj}\chi_{Hk}(t)\exp\left(-\int_t^T a_{Hk}(u)\,du\right)\chi_{Hj}(t)\exp\left(-\int_t^T a_{Hj}(u)\,du\right)dt$$
$$+\sum_{m=1}^{M} Var\left(\gamma_{mt} b_m(T)\exp\left(-\int_t^T b_m(u)\,du\right)dN_{mt}\gamma_{mt} b_m(T)\exp\left(-\int_t^T b_m(u)\,du\right)dN_{mt}\right)$$

and

$$V\,\mathrm{ar}\left(d\left(\ln C_t\right)\right) = \left(\sum_{k=1}^{K}\sum_{j=1}^{K} \rho_{HkHj}\left(\eta_{Hk}(t)+\chi_{Hk}(t)\right)\left(\eta_{Hj}(t)+\chi_{Hj}(t)\right)\right)dt$$
$$+\sum_{m=1}^{M} Var\left(\gamma_{mt}dN_{mt}\gamma_{mt}dN_{mt}\right) \qquad \text{(equation 3.7)}$$

Proof : Immediate from equations 3.4 and 3.6 •

Remark 3.10 : Now it is clear that $a_{Hk}(t)$, for each k, $k=1,2,...,K$, is playing the role of a mean reversion rate. We would therefore expect each $a_{Hk}(t)$ to be non-negative for all k and for all t and at least one of them to be strictly positive. We also require the jump decay coefficient functions $b_m(u)$, for each m to be non-negative. When this is the case, it is easy to see that the correlation, $correl\left(d\varepsilon(t,T),d\left(lnC_t\right)\right)$, between the log of the value of the commodity and the instantaneous futures convenience forward rate will always lie between zero and unity, provided that the correlation matrix ρ_{HkHj} is positive definite. This positive correlation is in line with the empirical evidence noted in section 1.

Remark 3.11 : Furthermore, perfect positive correlation would only occur in the case that our model reduces to a one-factor model (either diffusion or jump).

Remark 3.12 : A correlation of zero only happens when $a_{Hk}(t)\equiv 0$, for all k, $k=1,2,...,K$, and $b_m(t)\equiv 0$, for all m, $m=1,...,M$. However, inspection of the SDEs for the fictitious futures convenience yield bond price, the instantaneous futures convenience yield forward rate and the instantaneous futures convenience yield short rate show that if these latter are both true, then all these variables ie the fictitious futures convenience yield bond price, the instantaneous futures convenience yield forward rate and the instantaneous futures convenience yield short rate would be deterministic. That is, in this special case, they would have no diffusion volatility and no jumps.

Remark 3.13 : In a sense, it is the presence of non-zero mean reversion rate functions (ie $a_{Hk}(t)$) which makes futures convenience yields have a diffusion volatility and it is the existence of non-zero jump decay coefficient functions (ie $b_m(t)$) which makes futures convenience yields have jumps. In the special case that, for all m, the jump decay coefficient functions are all identically equal to zero (for example when all satisfy assumption 2.2), futures convenience yields have no jumps. Of course, this is intuitive, in view of equations 2.7, 2.8 and 2.9, since in this case when there are jumps there is a parallel shift in the log of the futures commodity prices across different tenors.

Remark 3.14 : Note that (using equation 3.7) the volatility of the value of the commodity, at time t, does not depend on the volatility of bond prices or interest-rates, nor does it depend on $a_{Hk}(t)$, for any k, $k=1,2,...,K$ nor on $b_m(t)$, for any m, $m=1,...,M$. This is entirely expected and in line with the intuition behind the construction of our model.

Remark 3.15 : Note that the volatility of the value of the commodity, at time t, depends on $\eta_{Hk}(t)$ but neither the volatility of the fictitious futures convenience yield bond price $P_\varepsilon(t,T)$ nor the volatility of the instantaneous futures convenience yield forward rate $\varepsilon(t,T)$, at time t, depend on $\eta_{Hk}(t)$, for any k (although their drift terms do).

The following proposition provides further insight into our model because it shows that the log of the value of the commodity exhibits mean reversion.

Proposition 3.16 : The log of the value of the commodity is a mean-reverting stochastic process whose SDE is of the form:

$$d(\ln C_t)=a_{H1}(t)(\Lambda(t_0,t,\ln H(t_0,t))-(\ln C_t))dt+\sum_{k=1}^{K}\sigma_{Hk}(t,t)dz_{Hk}(t)$$

$$+\sum_{m=1}^{M}\gamma_{mt}dN_{mt}-\left(\sum_{m=1}^{M}e_m\left(t,t\right)\right)dt \qquad \text{(equation 3.8)}$$

where $\Lambda\left(t_0,t,\ln H\left(t_0,t\right)\right)$ is defined by

$$a_{H1}(t)\Lambda\left(t_0,t,\ln H\left(t_0,t\right)\right)\equiv\frac{\partial\ln H\left(t_0,t\right)}{\partial t}+a_{H1}(t)\ln H\left(t_0,t\right)+\Psi\left(t_0,t\right)-\sum_{k=2}^{K}a_{Hk}(t)X_{Hk}(t)$$

$$+a_{H1}\left(t\right)\sum_{k=1}^{K}Y_{Hk}\left(t\right)+a_{H1}\left(t\right)\sum_{k=2}^{K}X_{Hk}\left(t\right)-\left(\frac{a_{H1}\left(t\right)}{\alpha_r}Y_P\left(t\right)\right)+\left(\left(\frac{a_{H1}\left(t\right)}{\alpha_r}-1\right)X_P\left(t\right)\right)$$

$$+a_{H1}\left(t\right)\sum_{m=1}^{M}X_{Nm}\left(t\right)-a_{H1}\left(t\right)\left(\sum_{m=1}^{M}\int_{t_0}^{t}e_m\left(s,t\right)ds\right)$$

$$-\sum_{m=1}^{M}b_m\left(t\right)X_{Nm}\left(t\right)-\left(\sum_{m=1}^{M}\int_{t_0}^{t}\frac{\partial e_m\left(s,t\right)}{\partial t}ds\right) \qquad \text{(equation 3.9)}$$

where $\Psi\left(t_0,t\right)$ is a deterministic function which depends only on t_0 and t (whose exact form is easily obtained at the expense of some tedious algebra).

Proof: We use our expression for $C_t\equiv H\left(t,t\right)$ (equation 2.22), and take logarithms, and our SDE for $d\left(\ln C_t\right)$ (equation 2.14), together with equation 2.21, to eliminate one of the state variables $X_{Hk}\left(t\right)$. The choice is arbitrary but to be definite, we eliminate $X_{H1}\left(t\right)$. We obtain equation 3.8 •

Remark 3.17 : This shows that $\ln C_t$ follows a mean reverting jump-diffusion process with a long run mean reversion level of $\Lambda\left(t_0,t,\ln H\left(t_0,t\right)\right)$. But we can see that $\Lambda\left(t_0,t,\ln H\left(t_0,t\right)\right)$ is stochastic and is itself also a mean reverting jump-diffusion process. In other words, the log of the value of the commodity follows a mean reverting jump-diffusion process whose long run mean reversion level is also a mean reverting jump-diffusion process.

Remark 3.18 : Note also how $X_P\left(t\right)$ and $Y_P\left(t\right)$ appear in equation 3.9. This shows that stochastic interest-rates can also contribute to this mean reversion. This is especially intuitive if risk-free interest-rates are negatively correlated with the Brownian motions driving the value of the commodity. If, for example, (note the form of equation 3.4) the Brownian motions driving the value of the commodity increases it, then the risk-free short rate will tend to decrease, which will tend to reduce the drift term on the SDE for the value of the commodity, which will then, ceterus paribus, tend to cause the value of the commodity to drift down. The reverse argument also holds. If the Brownian motions driving the value of the commodity decrease it, then the risk-free short rate will tend to increase, which will tend to increase the drift term on the SDE for the value of the commodity, which will then, ceterus paribus, tend to cause the value of the commodity to drift up.

Remark 3.19 : We stress again that the fictitious futures convenience yield bond price is a mathematical construction. We do not assume that such a bond really exists. Note also that in our model we have not had to make any additional assumptions about the nature of the stochastic evolution of instantaneous futures convenience yield forward rates or the futures convenience yield short rate. Their dynamics arise naturally from the assumption of the dynamics of futures commodity prices in equation 2.10.

Remark 3.20 : Note also that we have not had to make any assumptions about the nature of the market price of risk associated with fictitious futures convenience yield bond prices, instantaneous futures convenience yield forward rates or the futures convenience yield short rate.

Remark 3.21 : How can we summarise this model?
We have a multi-factor jump-diffusion model. The value of the commodity is driven by the K Brownian motions ($dz_{Hk}(t)$) plus the M Poisson processes (dN_{mt}). The fictitious futures convenience yield bond price and the instantaneous futures convenience yield forward rate are also driven by the same Brownian motions and (except in the special case that $b_m(t) \equiv 0$ for all m) the same Poisson processes. The log of the value of the commodity follows a mean reverting jump-diffusion process which mean reverts to a mean reversion level which is itself a mean reverting jump-diffusion process. It is noteworthy that stochastic interest-rates can also contribute to the mean reversion process. Futures commodity prices are also driven by the same K Brownian motions plus the Brownian motion driving interest-rates and bond prices plus the M Poisson processes. The correlation between the log of the value of the commodity and the instantaneous futures convenience yield forward rate is positive.

4. Monte Carlo simulation

In this section, we show how we can simulate futures commodity prices. The key to this will be to simulate the state variables since then we can use equation 2.23.

Monte Carlo simulation of the diffusion state variables is straightforward (see Babbs (1990), Dempster and Hutton (1997) or Glasserman (2004)). So now we examine how we can simulate the jump state variables, $X_{Nm}(t)$.

Recall that we make no assumptions about the spot jump amplitudes γ_{mt}, other than that, for each m, they satisfy either assumption 2.1 or they satisfy assumption 2.2. Although we index γ_{mt} with t, these assumptions mean that the outcomes of γ_{mt} do not depend on t.

Firstly, for future reference, we define, for each m,

$$\phi_m(t,T) \equiv \exp\left(-\int_t^T b_m(u)\,du\right). \qquad \text{(equation 4.1)}$$

Recall, that for each m, $m = 1,\ldots,M$, N_{mt} has a Poisson distribution with intensity rate $\lambda_m(t)$. The process starts at zero ie $N_{mt_0} \equiv 0$ and every time a jump occurs, the process increments by one. The process has independent increments.

The expected number of jumps in N_{mt} over the time period t_0 to t is $\int_{t_0}^{t} \lambda_m(u)\,du$.

Now, by the definition of a non-homogenous Poisson process, the probability that there are n_m jumps in the Poisson process N_{mt} in the time period t_0 to t is:

$$\Pr\left(N_{mt}=n_m\right)=\exp\left(-\int_{t_0}^{t}\lambda_m\left(u\right)du\right)\frac{\left[\int_{t_0}^{t}\lambda_m\left(u\right)du\right]^{n_m}}{n_m!} \qquad \text{(equation 4.2)}$$

We now state a very useful mathematical proposition.

Proposition 4.1 : Suppose that we know that there have been n_m jumps between time t_0 and time t. Write the arrival times of the jumps as $S_{1m},S_{2m},...,S_{n_m m}$. The conditional joint density function of the arrival times, when the arrival times are viewed as unordered random variables, conditional on $N_{mt}=n_m$ is:

$$\Pr\left(S_{1m}=s_{1m}\,\&\,S_{2m}=s_{2m}\,\&\ldots\&\,S_{n_m m}=s_{n_m m}\mid N_{mt}=n_m\right)=$$
$$\frac{\left[\lambda_m\left(s_{1m}\right)\right]\left[\lambda_m\left(s_{2m}\right)\right]\ldots\left[\lambda_m\left(s_{n_m m}\right)\right]}{\left[\int_{t_0}^{t}\lambda_m\left(u\right)du\right]^{n_m}} \qquad \text{(equation 4.3)}$$

Proof : The above result is proved in, for example, Karlin and Taylor (1975) in the case that the intensity rate is constant and the extension to a time-dependent deterministic intensity rate is straightforward (and therefore the proof is omitted). •

This is an important result because now it is straightforward to simulate $X_{Nm}\left(t\right)$. Firstly, we simulate the number of jumps n_m up to time t. There are several ways, given a random number generator which produces random numbers uniform on $\left(0,1\right)$, to simulate the number of jumps, in a given time interval, of a non-homogenous Poisson process (for example, see Glasserman (2004)). Using equation 4.3, we can simulate the arrival times $S_{1m},S_{2m},...,S_{n_m m}$ of the n_m jumps between time t_0 and time t. (This is particularly straightforward if $\lambda_m\left(t\right)$ is constant since then the arrival times, conditional on n_m, are uniform on $\left(t_0,t\right)$).

Now note that equation 2.19, the definition of $X_{Nm}\left(t\right)$, implies that

$$X_{Nm}\left(t\right)=\sum_{i=1}^{n_m}\gamma_{mS_{im}}\exp\left(-\int_{S_{im}}^{t}b_m\left(u\right)du\right)=\sum_{i=1}^{n_m}\gamma_{mS_{im}}\phi_m\left(S_{im},t\right). \qquad \text{(equation 4.4)}$$

If $n_m=0$, then $X_{Nm}\left(t\right)=0$. We include this case in equation 4.4 by using the usual convention that a summation is zero if the upper index is strictly less than the lower index.
It only remains to simulate γ_{mt} (in the case of assumption 2.1, the jump sizes are known constants and, in the case of assumption 2.2, they are independent and identically distributed which means they do not depend on the arrival times) and then we obtain $X_{Nm}\left(t\right)$ from equation 4.4.

In order to simulate futures commodity prices, we also need the final deterministic term in equation 2.23.

For each m, using equation 2.11:

$$\exp\left(-\int_{t_0}^{t} e_m(s,T)\,ds\right)=\exp\left(-\int_{t_0}^{t}\lambda_m(s)\,E_{Nms}\left(\exp\left(\gamma_{ms}\exp\left(-\int_{s}^{T}b_m(u)\,du\right)\right)-1\right)ds\right)$$

(equation 4.5)

Note that the integral in equation 4.5 would, in general, have to be done numerically, but it is a simple one dimensional deterministic integral which can be pre-computed before entering the Monte Carlo simulation.

We will use the following proposition in section 5.

Proposition 4.2 :

$$Exp_{t_0}\left[\exp\left(\sum_{m=1}^{M}\left(\exp\left(-\int_{t}^{T}b_m(u)\,du\right)X_{Nm}(t)\right)-\sum_{m=1}^{M}\int_{t_0}^{t}e_m(s,T)\,ds\right)\right]=1$$

(equation 4.6)

Proof : Use equations 4.2 and 4.3 and standard results about conditional expectations. •

Remark 4.3 : Note (leaving aside the issue of any errors in the evaluation of the deterministic integral in equation 4.5), that there are no discretisation error biases in the simulation of futures commodity prices in our model as there might be in some models involving the simulation of non-Gaussian stochastic processes (for discussions on this topic, see Babbs (2002) or Glasserman (2004)).

5. Option pricing

Our aim in this section is to derive the prices of standard options, and to do so in a form suitable for rapid computation. The key to this will be the observation that, conditional on the number of jumps and their arrival times (and with a suitable assumption about the spot jump amplitudes), futures commodity prices are log-normally distributed, at which point familiar results come into play (see also Merton (1976) and Jarrow and Madan (1995)). We will derive the prices of standard European options on futures, futures-style options on futures, standard European options on the spot and standard European options on forward commodity prices. Later in this section, we will provide some numerical examples which illustrate our model. We will also show that we can rapidly (typically of the order of 1/50th of a second per option depending upon the required accuracy) compute the prices of standard options.

To achieve our goals, we will have to make an assumption about the distribution of the spot jump amplitudes γ_{mt}. There are three cases of interest that offer the prospect of tractability. The first is to assume the spot jump amplitudes are constants as in assumption 2.1. Assumption 2.2 can be split into two possible cases which give our second and third cases of interest. The second case is to assume that the γ_{mt} are discrete random variables with a finite (in practice, small) number of possible values. We will examine this case in section 6.2 where we will see it can be considered as a particular case of the first. The third case is to assume the spot jump amplitudes γ_{mt} are normally distributed. We will examine the first and the third cases in this section.

For each m :

In the case of assumption 2.1, the spot jump amplitudes are assumed to be equal to β_m, a constant.

In the case of assumption 2.2, the spot jump amplitudes are assumed to be normally distributed with mean β_m and standard deviation υ_m (and in this case $b_m(t) \equiv 0$).

Clearly in the case of assumption 2.2, $\exp\left(\gamma_{ms}\exp\left(-\int_s^T b_m(u)\,du\right)\right)$ is log-normally distributed and using standard results for the expectation of the exponential of a normally distributed random variable we have (putting $b_m(t) \equiv 0$)

$$E_{Nms}\left(\exp\left(\gamma_{ms}\exp\left(-\int_s^T b_m(u)\,du\right)\right)\right) = \exp\left(\beta_m + \frac{1}{2}\upsilon_m^2\right) \qquad \text{(equation 5.1)}$$

A generic option pricing formula:

Our aim is to value, at time t, a European (non-path-dependent) option, maturing at time T_1, written on the futures commodity price, where the futures contract matures at time T_2 and $T_2 \geq T_1 \geq t$.

Conditional on the number of jumps n_m, $m = 1,...,M$, in the time period t to T_1, and the arrival times $s_{1m}, s_{2m},..., s_{n_m m}$, $m = 1,...,M$ of these jumps, then (using equations 4.1 and 4.4):

$$\exp\left(\left(\exp\left(-\int_{T_1}^{T_2} b_m(u)\,du\right)X_{Nm}(T_1)\right)\right) = \exp\left(\sum_{i=1}^{n_m}\beta_m\phi(s_{im}, T_2)\right) \qquad \text{(equation 5.2)}$$

in the case that assumption 2.1 is satisfied for this m;

or:

in the case that assumption 2.2 is satisfied for this m, $m = 1,...,M$, then

$\exp\left(\left(\exp\left(-\int_{T_1}^{T_2} b_m(u)\,du\right)X_{Nm}(T_1)\right)\right)$ is log-normally distributed with mean

$$\exp\left(\sum_{i=1}^{n_m}\left(\beta_m + \frac{1}{2}\upsilon_m^2\right)\right) = \exp\left(n_m\left(\beta_m + \frac{1}{2}\upsilon_m^2\right)\right) \qquad \text{(equation 5.3)}$$

Define the indicator functions, for each m, $m = 1,...,M$,

$1_{m(2.1)} = 1$ if assumption 2.1 is satisfied, for this m, and $1_{m(2.1)} = 0$ otherwise

and $1_{m(2.2)} = 1$ if assumption 2.2 is satisfied, for this m, and $1_{m(2.2)} = 0$ otherwise.

Proposition 5.1 : The futures commodity price $H(T_1,T_2)$ at time T_1 to time T_2, conditional on the futures price $H(t,T_2)$ at time t (where $t \le T_1 \le T_2$) and conditional on the number of jumps n_m, $m=1,...,M$, in the time period t to T_1, and the arrival times $s_{1m}, s_{2m},..., s_{n_m m}$, $m=1,...,M$ of these jumps, is log-normally distributed with mean

$$H(t,T_2)\exp\left(\left(\sum_{m=1}^{M}\left\{1_{m(2.1)}\sum_{i=1}^{n_m}\beta_m\phi(s_{im},T_2)+1_{m(2.2)}n_m\left(\beta_m+\frac{1}{2}\upsilon_m^2\right)\right\}\right)-\left(\sum_{m=1}^{M}\int_t^{T_1}e_m(s,T_2)\,ds\right)\right)$$

$$=H(t,T_2)V(t,T_1;n_m;T_2,M) \qquad \text{(equation 5.4)}$$

where

$$V(t,T_1;n_m;T_2,M)\equiv$$
$$\exp\left(\left(\sum_{m=1}^{M}\left\{1_{m(2.1)}\sum_{i=1}^{n_m}\beta_m\phi(s_{im},T_2)+1_{m(2.2)}n_m\left(\beta_m+\frac{1}{2}\upsilon_m^2\right)\right\}\right)-\left(\sum_{m=1}^{M}\int_t^{T_1}e_m(s,T_2)\,ds\right)\right) \qquad \text{(equation 5.5)}$$

And where:

In the case of assumption 2.1 being satisfied for a given m,

$$\exp\left(-\int_t^{T_1}e_m(s,T_2)\,ds\right)=\exp\left(-\int_t^{T_1}\lambda_m(s)\left(\exp\left(\phi_m(s,T_2)\beta_m\right)-1\right)ds\right) \qquad \text{(equation 5.6)}$$

And in the case of assumption 2.2 being satisfied for a given m,

$$\exp\left(-\int_t^{T_1}e_m(s,T_2)\,ds\right)=\exp\left(-\left\{\exp\left(\beta_m+\frac{1}{2}\upsilon_m^2\right)-1\right\}\left[\int_t^{T_1}\lambda_m(u)\,du\right]\right) \qquad \text{(equation 5.7)}$$

Proof: Equation 5.4 follows immediately from equation 2.23, taken together with equations 5.2 and 5.3. Equations 5.6 and 5.7 use the definitions in equations 4.1 and 2.11. •

Remark 5.2 : Note that, in general, it would be necessary to compute the integral in equation 5.6 numerically.

The integral of the instantaneous variance of log of $H(s,T_2)$ at time s, conditional on the number of jumps n_m, $m=1,...,M$, in the time period t to T_1, and the arrival times $s_{1m}, s_{2m},..., s_{n_m m}$, $m=1,...,M$ of these jumps, is $\Sigma^2(t,T_1,T_2,M)$ where

$$\Sigma^2(t,T_1,T_2,M)\equiv\int_t^{T_1}\left(\sum_{k=1}^{K}\sigma_{Hk}^2(s,T_2)+\sigma_P^2(s,T_2)-2\sum_{k=1}^{K}\rho_{PHk}\sigma_P(s,T_2)\sigma_{Hk}(s,T_2)\right)ds$$
$$+\int_t^{T_1}\left(\sum_{k=1}^{K}\sum_{j=1}^{k-1}2\rho_{HkHj}\sigma_{Hk}(s,T_2)\sigma_{Hj}(s,T_2)\right)ds+\sum_{m=1}^{M}\left(n_m\upsilon_m^2 1_{m(2.2)}\right) \qquad \text{(equation 5.8)}$$

Consider a non-path-dependent European option written on the futures commodity price. The option matures at time T_1 and the futures contract matures at time T_2. Let the payoff of the option at time T_1 be $D\left(H\left(T_1,T_2\right)\right)$ for some function $D(\bullet)$.

Conditional on the number of jumps n_m, $m=1,...,M$ and the arrival times $s_{1m},s_{2m},...,s_{n_m m}$, $m=1,...,M$ of these jumps, the value of the option at time t is (where $t \le T_1 \le T_2$):

$$Exp_t\left[\exp\left(-\int_t^{T_1} r(u)\,du\right)D\left(H\left(T_1,T_2\right)\right) \mid n_m, s_{1m}, s_{2m},...,s_{n_m m}; m=1,...,M\right]$$

(equation 5.9)

Remark 5.3 : In view of proposition 5.1, given the payoff $D\left(H\left(T_1,T_2\right)\right)$ of the option at time T_1, we will be able to use standard results (for log-normally distributed prices), together with equations 5.4, 5.5 and 5.8, to calculate the expectation in equation 5.9.

We define, for each m,

$$Q_m\left(t,T_1;n_m\right) \equiv \exp\left(-\int_t^{T_1}\lambda_m(u)\,du\right)\frac{\left(\int_t^{T_1}\lambda_m(u)\,du\right)^{n_m}}{n_m!} \qquad \text{(equation 5.10)}$$

Note that $Q_m\left(t,T_1;n_m\right)$ is just the probability that there are n_m jumps in the Poisson process N_{mt} in the time period t to T_1, for each m, $m=1,...,M$.

Proposition 5.4 : The price of the option at time t is:

$$\sum_{n_1=0}^{n_1=\infty}\sum_{n_2=0}^{n_2=\infty}...\sum_{n_M=0}^{n_M=\infty} Q_1\left(t,T_1;n_1\right)Q_2\left(t,T_1;n_2\right)...Q_M\left(t,T_1;n_M\right)$$
$$\int_t^{T_1}\int_t^{T_1}...\int_t^{T_1} Exp_t\left[\exp\left(-\int_t^{T_1} r(u)\,du\right)D\left(H\left(T_1,T_2\right)\right) \mid n_m, s_{1m}, s_{2m},...,s_{n_m m}; m=1,...,M\right]$$
$$\prod_{m=1}^{M}\frac{\left[\lambda_m\left(s_{1m}\right)\right]\left[\lambda_m\left(s_{2m}\right)\right]...\left[\lambda_m\left(s_{n_m m}\right)\right]}{\left[\int_t^{T_1}\lambda_m(u)\,du\right]^{n_m}} ds_{11}ds_{21}...ds_{n_1 1}ds_{12}ds_{22}...ds_{n_2 2}...ds_{1M}ds_{2M}...ds_{n_M M}$$

(equation 5.11)

Proof: It follows immediately from the results of section 4 (in particular equation 4.3), equations 5.9 and 5.10 and standard results about conditional expectations. ●

Remark 5.5 : Note that in the special case that for a given m, $m=1,...,M$, the spot jump amplitudes satisfy assumption 2.2, the integral over the arrival times of the Poisson jumps will be simplified as the integrand becomes independent of the arrival times of that given Poisson process.

Remark 5.6 : Further, in the special case that assumption 2.2 is satisfied for all m , $m = 1,...,M$, the option price at time t is simplified to:

$$\sum_{n_1=0}^{n_1=\infty} \sum_{n_2=0}^{n_2=\infty} ... \sum_{n_M=0}^{n_M=\infty} Q_1(t,T_1;n_1) Q_2(t,T_1;n_2) ... Q_M(t,T_1;n_M)$$

$$Exp_t \left[\exp\left(-\int_t^{T_1} r(u)\, du \right) D\big(H(T_1,T_2)\big) \mid n_m; m = 1,...,M \right] \qquad \text{(equation 5.12)}$$

Remark 5.7 : Furthermore, in this last special case, $V(t,T_1;n_m;T_2,M)$ also simplifies to (using equations 5.3 and 5.7):

$$V(t,T_1;n_m;T_2,M) =$$

$$\prod_{m=1}^{M} \exp\left(-\left\{ \exp\left(\beta_m + \frac{1}{2}\upsilon_m^2 \right) - 1 \right\} \left[\int_t^{T_1} \lambda_m(u)\, du \right] \right) \exp\left(n_m \left(\beta_m + \frac{1}{2}\upsilon_m^2 \right) \right)$$

(equation 5.13)

Using equations 5.4, 5.5, 5.8 and 5.11, we are now in a position to write down the prices of various standard options. Our specific option pricing formulae will come from substituting a specific form for equation 5.9 into equation 5.11. We state the results without proof but for full details and methodologies, see Merton (1973),(1976), Babbs (1990), Amin and Jarrow (1991), Duffie and Stanton (1992), Jamshidian (1993), Jarrow and Madan (1995) and especially Miltersen and Schwartz (1998).

Standard European Options on Futures:

Suppose that we wish to value at time t a standard European (call or put) option on the futures commodity price. The option matures at time T_1 and the futures contract matures at time T_2, where $T_2 \geq T_1$. The payoff of the option at time T_1 is $\max(\eta(H(T_1,T_2) - K),0)$ where K is the strike of the option and $\eta = 1$ if the option is a call and $\eta = -1$ if the option is a put.

The price of the option at time t is:

$$\sum_{n_1=0}^{n_1=\infty} \sum_{n_2=0}^{n_2=\infty} ... \sum_{n_M=0}^{n_M=\infty} Q_1(t,T_1;n_1) Q_2(t,T_1;n_2) ... Q_M(t,T_1;n_M) \eta P(t,T_1)$$

$$\int_t^{T_1}\int_t^{T_1} ... \int_t^{T_1} \left[H(t,T_2) V(t,T_1;n_m;T_2,M) \exp\left(\int_t^{T_1} A(s,T_1,T_2)\, ds \right) N(\eta d_1) - KN(\eta d_2) \right]$$

$$\prod_{m=1}^{M} \frac{[\lambda_m(s_{1m})][\lambda_m(s_{2m})]...[\lambda_m(s_{n_m m})]}{\left[\int_t^{T_1} \lambda_m(u)\, du \right]^{n_m}} ds_{11} ds_{21} ... ds_{n_1 1} ds_{12} ds_{22} ... ds_{n_2 2} ... ds_{1M} ds_{2M} ... ds_{n_M M}$$

(equation 5.14)

where

$$A(s,T_1,T_2) \equiv \left(\sum_{k=1}^{K} \rho_{PHk} \sigma_P(s,T_1) \sigma_{Hk}(s,T_2) \right) - \sigma_P(s,T_1) \sigma_P(s,T_2),$$

$$d_1 \equiv \frac{\ln\left(H(t,T_2)V(t,T_1;n_m;T_2,M)/K\right) + \left(\int_t^{T_1} A(s,T_1,T_2)\,ds\right) + \frac{1}{2}\Sigma^2(t,T_1,T_2,M)}{\Sigma(t,T_1,T_2,M)},$$

$$d_2 \equiv d_1 - \Sigma(t,T_1,T_2,M)$$

and $V(t,T_1;n_m;T_2,M)$ is as in equation 5.5 and $\Sigma(t,T_1,T_2,M)$ is as in equation 5.8

We recall in the above formula that $P(t,T_1)$ is the price of a zero coupon bond, at time t, maturing at time T_1, ie at option maturity. In the special case that assumption 2.2 is satisfied for all m, $m = 1,...,M$, the option price formula simplifies in view of equations 5.12 and 5.13.

Futures-style Options on Futures:

Suppose that we wish to value at time t a futures-style option (call or put) on the futures commodity price. Futures-style options are traded on some exchanges. The key point about futures-style options is that they are similar to futures contracts in that they go undergo continuous resettlement (in practice, daily resettlement) with a mark-to-market procedure and, as with futures contracts, there is no initial cost in buying a futures-style option. We assume that the futures-style option matures at time T_1 and the futures contract matures at time T_2, where $T_2 \geq T_1$. The futures-style option price (ie its delivery value) at time T_1 is $\max(\eta(H(T_1,T_2) - K),0)$ where K is the strike of the option and $\eta = 1$ if the option is a call and $\eta = -1$ if the option is a put. However, the gains and losses of the futures-style option are resettled continuously (in practice, daily) during the life of the futures-style option contract. It can be shown (see Merton (1990), Duffie (1996) or Duffie and Stanton (1992)) that the futures-style option price, at time t, is the price of a standard (ie non-futures-style) option, at time t, which has a payoff of $\left\{ \exp\left(\int_t^{T_1} r(u)\,du \right) \left[\max\left(\eta\left(H(T_1,T_2) - K\right),0\right)\right] \right\}$ at time T_1. Hence the futures-style option price, at time t, is:

$$Exp_t\left[\exp\left(-\int_t^{T_1} r(u)\,du \right) \exp\left(\int_t^{T_1} r(u)\,du \right) \left[\max\left(\eta\left(H(T_1,T_2) - K\right),0\right)\right] \right]$$
$$= Exp_t\left[\max\left(\eta\left(H(T_1,T_2) - K\right),0\right)\right]$$

Hence, we can show that the futures-style option price at time t is:

$$\sum_{n_1=0}^{n_1=\infty}\sum_{n_2=0}^{n_2=\infty}...\sum_{n_M=0}^{n_M=\infty} Q_1\left(t,T_1;n_1\right)Q_2\left(t,T_1;n_2\right)...Q_M\left(t,T_1;n_M\right)$$

$$\int_t^{T_1}\int_t^{T_1}...\int_t^{T_1}\eta\left[H\left(t,T_2\right)V\left(t,T_1;n_m;T_2,M\right)N\left(\eta d_1\right)-KN\left(\eta d_2\right)\right]$$

$$\prod_{m=1}^{M}\frac{\left[\lambda_m\left(s_{1m}\right)\right]\left[\lambda_m\left(s_{2m}\right)\right]...\left[\lambda_m\left(s_{n_m m}\right)\right]}{\left[\int_t^{T_1}\lambda_m\left(u\right)du\right]^{n_m}}ds_{11}ds_{21}...ds_{n_1 1}ds_{12}ds_{22}...ds_{n_2 2}...ds_{1M}ds_{2M}...ds_{n_M M}$$

(equation 5.15)

where

$$d_1\equiv\frac{\ln\left(H\left(t,T_2\right)V\left(t,T_1;n_m;T_2,M\right)/K\right)+\frac{1}{2}\Sigma^2\left(t,T_1,T_2,M\right)}{\Sigma\left(t,T_1,T_2,M\right)},$$

$$d_2\equiv d_1-\Sigma\left(t,T_1,T_2,M\right)$$

and $V\left(t,T_1;n_m;T_2,M\right)$ is as in equation 5.5 and $\Sigma\left(t,T_1,T_2,M\right)$ is as in equation 5.8.

Remark 5.8 : Note that the price of a zero coupon bond does not appear in equation 5.15.

Remark 5.9 : It can be shown (using the methods of Merton (1973), Duffie (1996), Duffie and Stanton (1992) and Jamshidian (1993)) that it is never optimal to exercise American futures-style options on futures prices before maturity which means that European futures-style options on futures prices and American futures-style options on futures prices always have the same price (this applies respectively to both calls and puts). Hence equation 5.15 is equally valid for both European and American futures-style options on futures prices.

Standard European Options on the spot:

Suppose that we wish to value at time t a standard European (call or put) option on the spot commodity price. The option matures at time T_1. The payoff of the option at time T_1 is $\max\left(\eta\left(C_{T_1}-K\right),0\right)$ where K is the strike of the option and $\eta=1$ if the option is a call and $\eta=-1$ if the option is a put. Note $C_{T_1}=F\left(T_1,T_1\right)$.

The price of the option at time t is:

$$\sum_{n_1=0}^{n_1=\infty}\sum_{n_2=0}^{n_2=\infty}\cdots\sum_{n_M=0}^{n_M=\infty} Q_1(t,T_1;n_1)Q_2(t,T_1;n_2)\ldots Q_M(t,T_1;n_M)\eta P(t,T_1)$$
$$\int_t^{T_1}\int_t^{T_1}\cdots\int_t^{T_1}\left[F(t,T_1)V(t,T_1;n_m;T_1,M)N(\eta d_1)-KN(\eta d_2)\right]$$
$$\prod_{m=1}^{M}\frac{\left[\lambda_m(s_{1m})\right]\left[\lambda_m(s_{2m})\right]\ldots\left[\lambda_m\left(s_{n_m m}\right)\right]}{\left[\int_t^{T_1}\lambda_m(u)\,du\right]^{n_m}}ds_{11}ds_{21}\ldots ds_{n_1 1}ds_{12}ds_{22}\ldots ds_{n_2 2}\ldots ds_{1M}ds_{2M}\ldots ds_{n_M M}$$

(equation 5.16)

where

$$d_1\equiv\frac{\ln\left(F(t,T_1)V(t,T_1;n_m;T_1,M)/K\right)+\frac{1}{2}\Sigma^2(t,T_1,T_1,M)}{\Sigma(t,T_1,T_1,M)},$$

$$d_2\equiv d_1-\Sigma(t,T_1,T_1,M)$$

and $V(t,T_1;n_m;T_1,M)$ is obtained from equation 5.5 and $\Sigma(t,T_1,T_1,M)$ is obtained from equation 5.8 (with $T_2\equiv T_1$).

Standard European Options on Forwards:

Suppose that we wish to value at time t a standard European (call or put) option on the forward commodity price. The option matures at time T_1 and the forward price is to time T_2, where $T_2\geq T_1$. We denote the strike of the option by K and we write $\eta=1$ if the option is a call and $\eta=-1$ if the option is a put.

There are two possible payoffs: (Note these are options on the forward commodity price and are not to be confused with the way that some options in the commodities markets actually work where the deliverable is a strip of forward prices over a period of time).

We consider first the case where the payoff of the option at time T_1 is $\max(\eta(F(T_1,T_2)-K),0)$.

The price of the option at time t is:

$$\sum_{n_1=0}^{n_1=\infty}\sum_{n_2=0}^{n_2=\infty}\cdots\sum_{n_M=0}^{n_M=\infty} Q_1(t,T_1;n_1)Q_2(t,T_1;n_2)\ldots Q_M(t,T_1;n_M)\eta P(t,T_1)$$

$$\int_t^{T_1}\int_t^{T_1}\cdots\int_t^{T_1}\left[F(t,T_2)V(t,T_1;n_m;T_2,M)\exp\left(\int_t^{T_1}B(s,T_1,T_2)\,ds\right)N(\eta d_1)-KN(\eta d_2)\right]$$

$$\prod_{m=1}^{M}\frac{\left[\lambda_m(s_{1m})\right]\left[\lambda_m(s_{2m})\right]\ldots\left[\lambda_m\left(s_{n_m m}\right)\right]}{\left[\int_t^{T_1}\lambda_m(u)\,du\right]^{n_m}}ds_{11}ds_{21}\ldots ds_{n_1 1}ds_{12}ds_{22}\ldots ds_{n_2 2}\ldots ds_{1M}ds_{2M}\ldots ds_{n_M M}$$

(equation 5.17)

where

$$B(s,T_1,T_2)\equiv\left(\sum_{k=1}^{K}\rho_{PHk}\left(\sigma_P(s,T_1)-\sigma_P(s,T_2)\right)\sigma_{Hk}(s,T_2)\right)$$
$$-\left(\sigma_P(s,T_1)-\sigma_P(s,T_2)\right)\sigma_P(s,T_2),$$

$$d_1\equiv\frac{\ln\left(F(t,T_2)V(t,T_1;n_m;T_2,M)/K\right)+\left(\int_t^{T_1}B(s,T_1,T_2)\,ds\right)+\frac{1}{2}\Sigma^2(t,T_1,T_2,M)}{\Sigma(t,T_1,T_2,M)},$$

$$d_2\equiv d_1-\Sigma(t,T_1,T_2,M)$$

and $V(t,T_1;n_m;T_2,M)$ is as in equation 5.5 and $\Sigma(t,T_1,T_2,M)$ is as in equation 5.8.

We consider secondly the case where the payoff of the option is also $\max(\eta(F(T_1,T_2)-K),0)$ but now the payoff occurs at time T_2. This means the payoff is the same as a payoff of $\max(\eta P(T_1,T_2)(F(T_1,T_2)-K),0)$ at time T_1.

The price of the option at time t is:

$$\sum_{n_1=0}^{n_1=\infty}\sum_{n_2=0}^{n_2=\infty}\cdots\sum_{n_M=0}^{n_M=\infty} Q_1(t,T_1;n_1)Q_2(t,T_1;n_2)\ldots Q_M(t,T_1;n_M)\eta P(t,T_2)$$

$$\int_t^{T_1}\int_t^{T_1}\cdots\int_t^{T_1}\left[F(t,T_2)V(t,T_1;n_m;T_2,M)N(\eta d_1)-KN(\eta d_2)\right]$$

$$\prod_{m=1}^{M}\frac{\left[\lambda_m(s_{1m})\right]\left[\lambda_m(s_{2m})\right]\ldots\left[\lambda_m\left(s_{n_m m}\right)\right]}{\left[\int_t^{T_1}\lambda_m(u)\,du\right]^{n_m}}ds_{11}ds_{21}\ldots ds_{n_1 1}ds_{12}ds_{22}\ldots ds_{n_2 2}\ldots ds_{1M}ds_{2M}\ldots ds_{n_M M}$$

(equation 5.18)

where

$$d_1 \equiv \frac{\ln\left(F\left(t,T_2\right)V\left(t,T_1;n_m;T_2,M\right)/K\right)+\frac{1}{2}\Sigma^2\left(t,T_1,T_2,M\right)}{\Sigma\left(t,T_1,T_2,M\right)},$$

$$d_2 \equiv d_1 - \Sigma\left(t,T_1,T_2,M\right)$$

and $V\left(t,T_1;n_m;T_2,M\right)$ is as in equation 5.5 and $\Sigma\left(t,T_1,T_2,M\right)$ is as in equation 5.8.

Note that, as with equation 5.14, equations 5.15, 5.16, 5.17 and 5.18 can all be simplified in the special case that assumption 2.2 is satisfied for all m, $m=1,...,M$, as indicated prior to equations 5.12 and 5.13.

Numerical examples and computational issues:

The above results are very useful as they also allow the possibility to calibrate the model through deriving implied parameters from the market prices of options. Clearly, for calibration purposes, rapid computation is important. We will now illustrate our model with a total of 8 numerical examples and also discuss computational issues surrounding the rapid computation of option prices using equations 5.14 to 5.18. The probabilities in the Poisson mass functions will rapidly tend to zero once the number of jumps is greater than the mean number of jumps. Therefore, computation times in the case when all the Poisson processes satisfy assumption 2.2 will typically be very small (at least when the number of Poisson processes is not too large). When all or some of the Poisson processes satisfy assumption 2.1, it is necessary to compute the integrals over the arrival times. The most appropriate method would seem to be to use Monte Carlo simulation of the arrival times (we stress only of the arrival times – not of the Poisson jumps nor the diffusion processes which can be done analytically). This is the method we use in the numerical examples below. Although this might sound computationally intensive, the simulation is just of the arrival times of the jumps. In many cases, the variation of the integral with different arrival times will be quite small leading to small standard errors. This might typically be the case for options which are deep in or out of the money or when the jump decay coefficient parameters ($b_m\left(t\right)$) are close to zero. In addition to minimise standard errors, we used the method of antithetic variates and we also used equation 4.6 as a control variate using the optimal-weighting/linear-regression methodology described, for example, in chapter 4 of Glasserman (2004). The option prices in tables 2, 5 and 8 (see our numerical examples below) were all computed using 1500 Monte Carlo simulations.

The deterministic integral in equation 5.6 was computed using the trapezium rule with 2500 points. Using a much larger number of points confirmed that the potential errors in the option prices in tables 2, 5 and 8 due to the approximation inherent in computing this integral were, in all cases, less than 0.000001 which is negligible compared to the standard errors reported. In the examples below, the summation over the Poisson probability mass functions was truncated when both the proportional and absolute convergence of the option price were less than 0.0001. Computations were performed on a desk-top p.c., running at 2.8 GHz, with Microsoft Windows 2000 Professional, with 1 Gb of RAM with a program written in Microsoft C++.

We now illustrate our model with a total of eight examples, labelled examples 1 to 8, the results of which are in tables 1 to 8 respectively. We will split them into two categories, examples 1 to 3 and then examples 4 to 8.
In all eight examples, we assume that the futures commodity prices to all maturities are 95 and the interest-rate yield curve is flat with a continuously compounded risk-free rate of 0.05 (as in Miltersen and Schwartz (1998)).

Although many of the parameters in our model can be time-dependent (and indeed it may be useful to allow for this to capture, for example, seasonality (see Miltersen (2003))), we will illustrate the model with constant parameters. In order to match the parameters of Miltersen and Schwartz (1998), whose set-up is slightly different to ours but entirely equivalent in the two factor pure-diffusion case, we choose to have two Brownian motions (in addition to the Brownian motion driving interest-rates) ie $K = 2$ and

$$\eta_{H1} = 0.266\,,\ \ \eta_{H2} = 0.249/1.045 \approx 0.23827751196\,,\ \ \chi_{H1} = 0.0\,,\ \ \chi_{H2} = -0.249/1.045$$
$$a_{H2} = 1.045$$
$$\sigma_r = 0.0096\,,\ \alpha_r = 0.2$$
$$\rho_{H1H2} = -0.805\,,\ \rho_{PH1} = -0.0964\,,\ \rho_{PH2} = 0.1243$$

Note the negative value of χ_{H2} is artificial in order to match the Miltersen and Schwartz (1998) data and could be made positive by combining η_{H1} and η_{H2} into one term and making consistent adjustments to the correlations in the obvious manner. Also, as remarked after equation 2.15, when calibrating our model, it would, in general, be necessary to put $\eta_{Hk}(t) \equiv 0$ for all k except one, in order to avoid a degeneracy.

We consider firstly examples 1 to 3. Example 1 is pure-diffusion and examples 2 and 3 are with jumps. The pure-diffusion example is effectively identical to that used in Miltersen and Schwartz (1998).

We value standard European call options (using equation 5.14) on futures contracts whose maturities are 0.125 years after the maturity of the option. We price options with strikes 75, 80, 95, 110, 115 and maturities equal to 0.25, 0.5, 0.75, 1, 2, 3 years (there are 30 options in total).

Example 1 :
In example 1, we price options in the pure-diffusion case (using equation 5.14 reduced to the no-jump case). The results are in table 1. Clearly the results are exactly as in table 1 of Miltersen and Schwartz (1998) (we have extra option maturities and extra strikes) since we have (albeit in a slightly different form) the same diffusion parameters. •

Now we introduce jump processes for examples 2 and 3 but keep the diffusion parameters as in example 1. The parameters of our processes are purely for illustration

Example 2 :
In example 2, we assume that there is one Poisson process, $M = 1$ and it satisfies assumption 2.1 and it has constant parameters:

$$\lambda_1 = 0.75\,,\ \beta_1 = 0.22\,,\ b_1 = 2.0$$

The parameters are only for illustration. The value of b_1 is roughly equivalent to the effect of a jump being "dampened" to approximately 37.8 % of the jump size over half a year which seems plausible.

Now we price options, using equation 5.14, with all the other parameters the same as in example 1. The results are in table 2. Also in the table are the corresponding standard errors (all are less than 0.0028) and the corresponding implied Black (1976) volatilities with[4] a price of 95.
The total computation time for all 30 options in this example was less than 0.51 seconds – or an average of less than 0.017 seconds per option. •

Example 3 :
In example 3, we assume that there are two Poisson processes, $M = 2$ and they both satisfy assumption 2.2, with parameters:

$$\lambda_1 = 0.75,\ \beta_1 = 0.22,\ \upsilon_1 = 0.01,\ b_1 = 0.0$$
$$\lambda_2 = 0.75,\ \beta_2 = -0.15,\ \upsilon_2 = 0.01,\ b_2 = 0.0$$

Again, the parameters are only for illustration. The intuition of the parameter values, loosely speaking, is to try to capture a commodity which can have upward jumps and also have downward jumps of slightly smaller size, the intensity rates of the two Poisson processes being equal.

Now we again price options, using equation 5.14, with the other parameters the same as in example 1. The results are in table 3. Also in the table are the corresponding implied Black (1976) volatilities with a price of 95. Since $b_1 = 0.0$ and $b_2 = 0.0$, there is no integration over the arrival times and hence computation times were negligible compared to those in example 2. •

It can be seen that in both examples with jumps (that is, examples 2 and 3), the model produces a volatility skew. The magnitude of the skew decreases with increasing option maturity which is typical for jump-diffusion processes.

Miltersen and Schwartz (1998) provide examples (see table 3 of their paper) of the impact of futures maturity on option prices.

We now consider a total of five more examples, labelled examples 4 to 8, which show the impact of futures maturity on option prices in our model. In all five of these examples, we use the same futures prices, interest-rates, number of Brownian motions and diffusion parameters as in examples 1 to 3. In each case, we price standard European call options (using equation 5.14) with one year to option maturity with the same strikes as before. We now consider three values of futures maturity, namely 1.125, 2, 3 years which therefore correspond to futures contracts maturing 0.125, 1, 2 years after option maturity. There are a total of 15 options.

Example 4 :
In example 4, we price options in the pure-diffusion case (using equation 5.14 reduced to the no-jump case). The prices coincide with the corresponding results in table 3 of Miltersen and Schwartz (1998) (again we have more options). Our results are in table 4. •

[4] We have used the futures price of 95 when calculating the implied Black (1976) volatilities for all our examples because this seems to be in line with the market convention. The market convention appears to effectively ignore the difference between forward prices and futures prices. However, in our model, interest-rates are stochastic and so it is necessary to distinguish between forward prices and futures prices. For illustration, with our diffusion parameters, a futures price of 95 on a futures contract with 3.125 years to maturity corresponds (using equation 6.29) to a forward price of 94.939. This is certainly not a large difference although it would be much larger for futures contracts with larger times to maturity. For example, on a contract with 12 years to maturity, a futures price of 95 corresponds to a forward price of 93.941 or a difference of more than 1.1%.

Example 5 :
In example 5, we price options with the same Poisson parameters as in example 2 ie we assume that there is one Poisson process, $M = 1$ and it satisfies assumption 2.1 and it has constant parameters:

$$\lambda_1 = 0.75 \,,\ \beta_1 = 0.22 \,,\ b_1 = 2.0$$

The results are in table 5. Also in the table are the corresponding standard errors (all are less than 0.0014) and the corresponding implied Black (1976) volatilities with a price of 95.
The total computation time for all 15 options in this example was less than 0.24 seconds – or an average of less than 0.016 seconds per option. •

Example 6 :
In example 6, we price options with the same Poisson parameters as in example 3 ie we assume that there are two Poisson processes, $M = 2$ and they both satisfy assumption 2.2, with parameters:

$$\lambda_1 = 0.75 \,,\ \beta_1 = 0.22 \,,\ \upsilon_1 = 0.01 \,,\ b_1 = 0.0$$
$$\lambda_2 = 0.75 \,,\ \beta_2 = -0.15 \,,\ \upsilon_2 = 0.01 \,,\ b_2 = 0.0$$

The results are in table 6. •

Example 7 :
In example 7, we price options with the same Poisson parameters as in examples 2 and 5 except now we set $b_1 = 0.0$ ie we assume that there is one Poisson process, $M = 1$ and it satisfies assumption 2.1 and it has constant parameters:

$\lambda_1 = 0.75 \,,\ \beta_1 = 0.22 \,,\ b_1 = 0.0$ (and also $\upsilon_1 = 0.0$ which is required for assumption 2.1)

The results are in table 7. Since $b_1 = 0.0$, there is no integration over the arrival times and hence computation times were negligible compared to those in example 5. •

Example 8 :
In example 8, we price options with the same Poisson parameters as in examples 2, 5 and 7 except now we set $b_1 = 4.0$ ie we assume that there is one Poisson process, $M = 1$ and it satisfies assumption 2.1 and it has constant parameters:

$$\lambda_1 = 0.75 \,,\ \beta_1 = 0.22 \,,\ b_1 = 4.0$$

The results are in table 8. Also in the table are the corresponding standard errors (all are less than or equal to 0.0008) and the corresponding implied Black (1976) volatilities with a price of 95.
The total computation time for all 15 options in this example was approximately the same as in example 5 ie an average of less than 0.016 seconds per option. •

In all cases, we see that the presence of jumps increases option prices relative to the no-jump case as proven in Merton (1976).

If we consider examples 4, 8, 5 and 7 in that order, we see an ordering of the option prices for a given strike, option maturity and futures maturity. Option prices in example 4 (no jumps) are less than or equal to those in example 8 (one Poisson process, $b_1 = 4.0$) which, in turn, are less than those in example 5 (one Poisson process, $b_1 = 2.0$) which, in turn, are less than those in example 7 (one Poisson process, $b_1 = 0.0$). These results are in line with the intuition behind the model. We can also see comparing the final lines (ie with futures maturity equal to three years) of tables 4 and 8 that the differences between the option prices, in the case of no jumps (example 4) and the case where

$b_1 = 4.0$ (example 8), are virtually negligible. This can be explained by the high value of the jump decay coefficient parameter ($b_1 = 4.0$) which causes a large "dampening" of the jumps. These examples also reinforce the remark made in Miltersen and Schwartz (1998) about the importance of futures maturity (not just option maturity) on option prices.

Our experience is that most standard (plain vanilla) options currently traded in the commodities markets have maturities which are usually less than two months (and often just a few weeks) before the maturity of the underlying futures contracts. As the commodity derivative markets expand, this may well change (compare the development of the interest-rate derivatives markets: at one time, caps were much more common than swaptions but now the situation is almost reversed), in which case our model may be particularly useful. Our model may also prove very useful for pricing complex commodity derivatives utilising the Monte Carlo simulation method described in section 4.

The computation times reported above were with code for which a reasonable but not exhaustive effort had been made to optimise for speed. It would probably be possible to speed the code up further. One simple method to improve the efficiency of the option price calculations above would be to use more (respectively fewer) Monte Carlo simulations for those options which generated the largest (respectively smallest) proportional standard errors, which in our examples are out-of-the-money (respectively in-the-money) call options. A possible topic for further research might be to further investigate variance reduction techniques (see, for example, Glasserman (2004)).

6. No-arbitrage, market completeness and incompleteness

In this section, we derive no-arbitrage conditions for our model. We will show that these conditions are closely linked to assumptions 2.1 and 2.2 which we made about the nature of the spot jump amplitudes and the jump decay coefficient functions. We will also explain the conditions in which our model leads to complete and incomplete markets, derive partial integro-differential equations satisfied by the price of derivatives and relate futures prices to forward prices.

We refer readers to Hoogland et al. (2001), Bjork et al. (1997), Babbs and Webber (1994), Jarrow and Madan (1995) and Runggaldier (2002). The above papers cover somewhat similar material albeit usually in models of the term structure of interest-rates. The paper by Hoogland et al. (2001) is closest to our model in this regard. Bjork et al. (1997) provide a very rigorous mathematical treatment.

Consider a market with frictionless continuous trading in a time interval $\left[t_0, T_{\max}\right]$, with $t_0 < T_{\max} < \infty$, which is a fixed time horizon long enough to contain the maturity times of all futures contracts and bonds of interest.

Let $\left(\Omega, \mathrm{F}, Q^w\right)$ be a probability space equipped with a filtration $\left\{\mathrm{F}_t\right\}_{t\in[t_0, T_{\max}]}$ satisfying the usual conditions. We identify Q^w as the "real **w**orld" probability measure – hence the superscript ^{w}.

Introduce $w_P\left(t\right)$ and $w_{Hk}\left(t\right)$, for each k, $k = 1, 2, ..., K$, which are Brownian motions under Q^w adapted to the filtration F and N_{mt}, for each m, $m = 1, ..., M$, which are Poisson processes with Q^w intensities $\lambda_m^w\left(t\right)$ which are also adapted to the filtration F. We assume that $\lambda_m^w\left(t\right)$ is a deterministic function of, at most, t and that it is strictly positive and bounded, for each m. We know that $N_{mt} - \int\limits_{t_0}^{t} \lambda_m^w\left(s\right) ds$ is a martingale (the compensated martingale) under Q^w.

We assume that interest-rates are stochastic and are as in the extended Vasicek model. Using the same notation as in equation 2.1, the dynamics of bond prices under Q^w are assumed to be of the form:

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)\,dt + \sigma_P(t,T)\,dw_P(t) \qquad \text{(equation 6.1)}$$

We assume that we have futures contracts of different maturities T_i. We assume that futures commodity prices under Q^w are of the form:

$$\frac{dH(t,T_i)}{H(t,T_i)} = \mu_i(t,\bullet)\,dt + \sum_{k=1}^{K} \sigma_{Hk}(t,T_i)\,dw_{Hk}(t) - \sigma_P(t,T_i)\,dw_P(t)$$
$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\phi_m(t,T_i)\right)-1\right)dN_{mt} \qquad \text{(equation 6.2)}$$

where $\mu_i(t,\bullet)$ is the drift term whose form is unknown. Note $\phi_m(t,T_i)$ is given by equation 4.1.

We would like to establish a suitable form for the drift term; ie a form for the drift term which is compatible with the absence of arbitrage, because, of course, under Q^w, futures commodity prices need not be martingales.

In section 6, we shall consider several possible forms for the nature of the spot jump amplitudes γ_{mt} introduced in section 2. The first is when the spot jump amplitudes are known and constant. We consider this case in subsection 6.1. The second case is when the spot jump amplitudes are discrete random variables with a finite (in practice, small) number of possible states. We consider this case in subsection 6.2. The third case is that the spot jump amplitudes γ_{mt} are continuous random variables which are independent and identically distributed. We consider this case in subsection 6.3. In section 5, we specialised this case to the case where they have a normal distribution but greater generality might be possible. In subsection 6.4, we consider the case where there is a mixture of the previous cases. In subsection 6.5, we will relate futures commodity prices and forward commodity prices.

6.1 Constant spot jump amplitudes

We assume in this section that the spot jump amplitudes are constant. We assume that our market is arbitrage-free and wish to establish a functional form for the drift which is consistent with this assumption.

We assume that we have $K+M+2$ futures contracts of $K+M+2$ different maturities T_i, $i=1,\ldots,K+M+2$, each of whose prices, under Q^w follow equation 6.2.

“Real world” drift in the absence of arbitrage:

We set up a portfolio Π of $K+M+2$ futures contracts consisting of x_i units of the futures contract maturing at time T_i, where $x_i \equiv x_i(t)$ are F previsible. It costs nothing to set up such a portfolio since there is no up-front cost in entering into a futures contract. The change in the value of the portfolio, $d\Pi$, in the time period t to $t+dt$ is:

$$d\Pi = \sum_{i=1}^{K+M+2} x_i dH(t,T_i)$$

Substituting from equation 6.1, this is:

$$d\Pi = \sum_{i=1}^{K+M+2} \left(x_i \left(\mu_i(t,\bullet)dt + \sum_{k=1}^{K} \sigma_{Hk}(t,T_i)dw_{Hk}(t) - \sigma_P(t,T_i)dw_P(t) \right) H(t,T_i) \right)$$
$$+ \sum_{i=1}^{K+M+2} \left(x_i \left(\sum_{m=1}^{M} \left(\exp\left(\gamma_{mt}\phi_m(t,T_i)\right) - 1 \right) dN_{mt} \right) H(t,T_i) \right) \qquad \text{(equation 6.3)}$$

In this section, the spot jump amplitudes γ_{mt}, for $m = 1,...,M$ are assumed constant. We seek to choose the portfolio weights x_i (not all zero) such that the portfolio is instantaneously risk-free. In this case, in the absence of arbitrage, since the portfolio cost nothing to enter into, its return in the period t to $t+dt$ must be zero.

But for this to be possible, it must be the case that, for each i, $\mu_i(t,\bullet)$ is of the form:

$$\mu_i(t,\bullet) = \sum_{k=1}^{K} \vartheta_k(t,\bullet)\sigma_{Hk}(t,T_i) - \vartheta_{K+M+1}(t,\bullet)\sigma_P(t,T_i)$$
$$+\sum_{m=1}^{M} \left(\vartheta_{m+K}(t,\bullet)\left(\exp\left(\gamma_{mt}\phi_m(t,T_i)\right) - 1 \right) \right) \qquad \text{(equation 6.4)}$$

for some $\vartheta_k(t,\bullet)$, where $\vartheta_k(t,\bullet)$, for each k, $k = 1,...,K+M+1$, are not dependent upon T_i and hence are not dependent upon any $H(t,T_i)$ or upon any $P(t,T_i)$. However $\vartheta_k(t,\bullet)$ may depend upon time t and possibly other variables.

Remark 6.1 : Equation 6.4 states the form that the drift term in the SDE for futures prices, under the “real world” measure Q^w, must have in order to be compatible with the assumption of no-arbitrage.

Remark 6.2 : It is clear that $\vartheta_k(t,\bullet)$ for each k, $k = 1,2,...,K$ are the market prices of risk associated with each $dw_{Hk}(t)$ and $\vartheta_{K+M+1}(t,\bullet)$ is the market price of risk associated with $dw_P(t)$.

Define $g_m(t,T_i) \equiv \exp\left(\gamma_{mt}\phi_m(t,T_i)\right) - 1$ (equation 6.5)

Then we can rewrite equation 6.2, our SDE for $H(t,T_i)$, in the form:

$$\frac{dH(t,T_i)}{H(t,T_i)} = \sum_{k=1}^{K} \sigma_{Hk}(t,T_i)\left(\vartheta_k(t,\bullet)dt + dw_{Hk}(t)\right) - \sigma_P(t,T_i)\left(\vartheta_{K+M+1}(t,\bullet)dt + dw_P(t)\right)$$

$$+\sum_{m=1}^{M} g_m(t,T_i)\left(dN_{mt} + \vartheta_{m+K}(t,\bullet)dt\right) \qquad \text{(equation 6.6)}$$

It is clear that the $\vartheta_{m+K}(t,\bullet)$, for each m, $m=1,...,M$, are playing a slightly analogous role for the Poisson processes as $\vartheta_k(t,\bullet)$ for each k, $k=1,2,...,K$ and $\vartheta_{K+M+1}(t,\bullet)$ play for the Brownian motions.

Derivative pricing:

Now let us consider a derivative or contingent claim written on these futures contracts.

We denote the price of the derivative at time t by $D(t)$. We suppose that it matures at time T. The derivative price $D(t) \equiv D(t, H_1, H_2, ..., H_{K+M+1}, P)$ where H_i is short-hand for $H_i \equiv H(t,T_i)$ and P is short-hand for the price of a bond. We will consider the case that the bond matures at time T, since the extension if it does not (or the price of the derivative depends on the prices of more than one bond) is straightforward.

By Ito's lemma, and using equation 6.6,

$$dD = D_t dt + \frac{1}{2}\sum_{i=1}^{K+M+1}\sum_{j=1}^{K+M+1} D_{H_iH_j} dH_i dH_j$$

$$+D_P dP + \frac{1}{2} D_{PP} dPdP + \sum_{j=1}^{K+M+1} D_{PH_j} dPdH_j$$

$$+\sum_{i=1}^{K+M+1} D_{H_i}\left(\sum_{k=1}^{K} \sigma_{Hk}(t,T_i)\left(\vartheta_k(t,\bullet)dt + dw_{Hk}(t)\right) - \sigma_P(t,T_i)\left(\vartheta_{K+M+1}(t,\bullet)dt + dw_P(t)\right)\right)H_i$$

$$+\sum_{i=1}^{K+M+1} D_{H_i}\left(\sum_{m=1}^{M}\left(\vartheta_{m+K}(t,\bullet)\, g_m(t,T_i)\, dt\right)\right)H_i$$

$$+\sum_{m=1}^{M}\left(D\left(\bar{g}_m\bar{H} + \bar{H}\right) - D(t)\right)dN_{mt} \qquad \text{(equation 6.7)}$$

where subscripts denote partial derivatives ie

$$D_t \equiv \frac{\partial D}{\partial t},\ D_{H_i} \equiv \frac{\partial D}{\partial H_i},\ D_{H_iH_j} \equiv \frac{\partial^2 D}{\partial H_i \partial H_j},\ D_P \equiv \frac{\partial D}{\partial P},\ D_{PP} \equiv \frac{\partial^2 D}{\partial P \partial P} \text{ and } D_{PH_j} \equiv \frac{\partial^2 D}{\partial P \partial H_j}$$

and $D\left(\bar{g}_m\bar{H} + \bar{H}\right)$ is a short-hand notation for the price of the derivative immediately after a jump in the Poisson process N_{mt}, for each m, if a jump occurred at time t,
and where (again using short-hand notation)

$$dPdP = \sigma_P^2(t,T)\, P^2(t,T)\, dt$$

$$dPdH_j = \left(\sum_{k=1}^{K} \rho_{PHk} \sigma_P (t,T) \sigma_{Hk} (t,T_j) - \sigma_P (t,T) \sigma_P (t,T_j) \right) P(t,T) H(t,T_j) dt$$

$$dH_i dH_j = \left\{ \sigma_P (t,T_i) \sigma_P (t,T_j) + \sum_{k=1}^{K} \sum_{l=1}^{K} \rho_{HkHl} \sigma_{Hk} (t,T_i) \sigma_{Hl} (t,T_j) \right\} H(t,T_i) H(t,T_j) dt$$
$$- \left\{ \sum_{l=1}^{K} \rho_{PHl} \sigma_P (t,T_i) \sigma_{Hl} (t,T_j) + \sum_{k=1}^{K} \rho_{PHk} \sigma_P (t,T_j) \sigma_{Hk} (t,T_i) \right\} H(t,T_i) H(t,T_j) dt$$

(equation 6.8)

Now, in the spirit of Merton (1973), let us form a portfolio Π consisting of one derivative, x_i units of futures contracts maturing at time T_i for each i, $i = 1,...,K+M+1$ and x_P bonds, where $x_i \equiv x_i(t)$ and $x_P \equiv x_P(t)$ are F previsible.
The portfolio is to be self-financing and is to require zero net aggregate investment. The value of the portfolio, at time t, is:

$$0 = D + x_P P \qquad \text{(equation 6.9)}$$

since there is no up-front cost in entering into futures contracts and the portfolio is constructed so as to have zero net aggregate investment.

The change in the value of the portfolio $d\Pi$ over the time period t to $t+dt$ is:

$$d\Pi = dD + \sum_{i=1}^{K+M+1} x_i dH_i + x_P dP ,$$

which from equations 6.6 and 6.7 is

$$d\Pi = D_t dt + \frac{1}{2} \sum_{i=1}^{K+M+1} \sum_{j=1}^{K+M+1} D_{H_i H_j} dH_i dH_j + \sum_{m=1}^{M} \left(D\left(\overline{g}_m \overline{H} + \overline{H} \right) - D(t) \right) dN_{mt}$$
$$+ D_P dP + \frac{1}{2} D_{PP} dPdP + \sum_{j=1}^{K+M+1} D_{PH_j} dPdH_j$$
$$+ \sum_{i=1}^{K+M+1} D_{H_i} \left(\left(\sum_{k=1}^{K} \sigma_{Hk} (t,T_i) \left(\vartheta_k (t,\bullet) dt + dw_{Hk}(t) \right) - \sigma_P (t,T_i) \left(\vartheta_{K+M+1} (t,\bullet) dt + dw_P(t) \right) \right) \right) H_i$$
$$+ \sum_{i=1}^{K+M+1} D_{H_i} \left(\sum_{m=1}^{M} \left(\vartheta_{m+K} (t,\bullet) g_m (t,T_i) dt \right) \right) H_i$$
$$+ \sum_{i=1}^{K+M+1} x_i \left(\sum_{k=1}^{K} \sigma_{Hk} (t,T_i) \left(\vartheta_k (t,\bullet) dt + dw_{Hk}(t) \right) - \sigma_P (t,T_i) \left(\vartheta_{K+M+1} (t,\bullet) dt + dw_P(t) \right) \right) H_i$$
$$+ \sum_{i=1}^{K+M+1} x_i \left(\sum_{m=1}^{M} \left(g_m (t,T_i) dN_{mt} + \vartheta_{m+K} (t,\bullet) g_m (t,T_i) dt \right) \right) H_i$$
$$+ x_P dP \qquad \text{(equation 6.10)}$$

which we can rewrite as

$$d\Pi = D_t dt + \frac{1}{2}\sum_{i=1}^{K+M+1}\sum_{j=1}^{K+M+1} D_{H_iH_j} dH_i dH_j - \sum_{m=1}^{M} \vartheta_{m+K}(t,\bullet)\left(D\left(\overline{g}_m\overline{H}+\overline{H}\right)-D(t)\right)dt$$
$$+\sum_{i=1}^{K+M+1} D_{H_i}\left(\sum_{m=1}^{M}\left(\vartheta_{m+K}(t,\bullet)\, g_m(t,T_i)\right)\right)H_i dt$$
$$+\frac{1}{2}D_{PP}dPdP + \sum_{j=1}^{K+M+1} D_{PH_j} dPdH_j$$
$$+\sum_{i=1}^{K+M+1} D_{H_i}\left(\left(\sum_{k=1}^{K}\sigma_{Hk}(t,T_i)\left(\vartheta_k(t,\bullet)dt + dw_{Hk}(t)\right) - \sigma_P(t,T_i)\left(\vartheta_{K+M+1}(t,\bullet)dt + dw_P(t)\right)\right)\right)H_i$$
$$+\sum_{i=1}^{K+M+1} x_i\left(\sum_{k=1}^{K}\sigma_{Hk}(t,T_i)\left(\vartheta_k(t,\bullet)dt + dw_{Hk}(t)\right) - \sigma_P(t,T_i)\left(\vartheta_{K+M+1}(t,\bullet)dt + dw_P(t)\right)\right)H_i$$
$$+\sum_{i=1}^{K+M+1} x_i\left(\sum_{m=1}^{M}\left(g_m(t,T_i)\left(dN_{mt} + \vartheta_{m+K}(t,\bullet)dt\right)\right)\right)H_i$$
$$+\sum_{m=1}^{M}\left(D\left(\overline{g}_m\overline{H}+\overline{H}\right)-D(t)\right)\left(dN_{mt}+\vartheta_{m+K}(t,\bullet)dt\right)$$
$$+\left(x_P + D_P\right)dP \qquad \text{(equation 6.11)}$$

Now suppose that we can find x_i, for each i, $i = 1,\ldots,K+M+1$, and x_P such that all the terms in $dw_P(t)$, $dw_{Hk}(t)$, for each k, and dN_{mt} for each m, vanish then the portfolio will be instantaneously risk-free.

We can choose $x_P = -D_P$, then the last line of equation 6.11 vanishes. If the derivative price is linear in P then the zero net aggregate condition given by equation 6.9 will automatically be satisfied.

Then, in order for the portfolio to be instantaneously risk-free, we need:

$$\sum_{i=1}^{K+M+1} D_{H_i}\left(\left(\sum_{k=1}^{K}\sigma_{Hk}(t,T_i)\left(\vartheta_k(t,\bullet)dt + dw_{Hk}(t)\right) - \sigma_P(t,T_i)\left(\vartheta_{K+M+1}(t,\bullet)dt + dw_P(t)\right)\right)\right)H_i$$
$$+\sum_{i=1}^{K+M+1} x_i\left(\sum_{k=1}^{K}\sigma_{Hk}(t,T_i)\left(\vartheta_k(t,\bullet)dt + dw_{Hk}(t)\right) - \sigma_P(t,T_i)\left(\vartheta_{K+M+1}(t,\bullet)dt + dw_P(t)\right)\right)H_i$$
$$+\sum_{i=1}^{K+M+1} x_i\left(\sum_{m=1}^{M}\left(g_m(t,T_i)\left(dN_{mt} + \vartheta_{m+K}(t,\bullet)dt\right)\right)\right)H_i$$
$$+\sum_{m=1}^{M}\left(D\left(\overline{g}_m\overline{H}+\overline{H}\right)-D(t)\right)\left(dN_{mt}+\vartheta_{m+K}(t,\bullet)dt\right) = 0 \qquad \text{(equation 6.12)}$$

We can view equation 6.12 as a linear system and, using matrix algebra, solve for x_i provided the square $K+M+1$ by $K+M+1$ matrix whose i^{th} column and j^{th} row, is $\sigma_{Hj}(t,T_i)$ if $1 \le j \le K$, is $g_{j-K}(t,T_i)$ if $K < j \le K+M$ and is $-\sigma_P(t,T_i)$ if $j = K+M+1$, is invertible. But of course, it will be invertible if the drift terms satisfy equation 6.4 for each i, $i = 1,\ldots,K+M+1$. So this means we can choose x_i as required ie such that the portfolio will be instantaneously risk-free. Since the portfolio cost nothing to enter into then, in the absence of arbitrage, its realised return in the period t to $t+dt$ must be zero. Hence:

$$0 = D_t dt + \frac{1}{2}\sum_{i=1}^{K+M+1}\sum_{j=1}^{K+M+1} D_{H_iH_j} dH_i dH_j - \sum_{m=1}^{M} \vartheta_{m+K}\left(t,\bullet\right)\left(D\left(\overline{g}_m\overline{H}+\overline{H}\right)-D\left(t\right)\right)dt$$
$$+\sum_{i=1}^{K+M+1} D_{H_i}\left(\sum_{m=1}^{M}\left(\vartheta_{m+K}\left(t,\bullet\right)g_m\left(t,T_i\right)\right)\right)H_i dt$$
$$+\frac{1}{2}D_{PP}dPdP + \sum_{j=1}^{K+M+1} D_{PH_j} dPdH_j \qquad \text{(equation 6.13)}$$

Remark 6.3 : As in the case when there are no jumps (ie a pure diffusion), the $\vartheta_k\left(t,\bullet\right)$ for $k=1,2,...,K$ and for $k=K+M+1$ cancel (ie the market prices of risk for the Brownian motions cancel). However, the $\vartheta_{m+K}\left(t,\bullet\right)$, for each m, do not cancel from equation 6.13.

Because we have eliminated the sources of risk in our portfolio hedging the derivative (or alternatively, we could have been replicating the derivative), we know that our market is complete. Therefore, we know (Harrison and Pliska (1981), Duffie (1996)) that there exists a unique equivalent martingale measure, which we denote by Q, under which futures prices are martingales.

Proposition 6.4 : The dynamics of futures prices, under Q, are:

$$\frac{dH\left(t,T_i\right)}{H\left(t,T_i\right)} = \sum_{k=1}^{K}\sigma_{Hk}\left(t,T_i\right)dz_{Hk}\left(t\right) - \sigma_P\left(t,T_i\right)dz_P\left(t\right)$$
$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\phi_m\left(t,T_i\right)\right)-1\right)dN_{mt} - \sum_{m=1}^{M}\lambda_m\left(t\right)\left(\exp\left(\gamma_{mt}\phi_m\left(t,T_i\right)\right)-1\right)dt$$

(equation 6.14)

where $z_P\left(t\right)$ and $z_{Hk}\left(t\right)$, for each k, $k=1,2,...,K$, defined by

$$z_P\left(t\right) = w_P\left(t\right) + \int_{t_0}^{t}\vartheta_{K+M+1}\left(s,\bullet\right)ds \qquad \text{(equation 6.15)}$$

and $z_{Hk}\left(t\right) = w_{Hk}\left(t\right) + \int_{t_0}^{t}\vartheta_k\left(s,\bullet\right)ds$ (equation 6.16)

are Brownian motions under Q and N_{mt}, for each m, $m=1,...,M$, are Poisson processes with Q intensity rates $\lambda_m\left(t\right)$, that is,

for each m, $m=1,...,M$, $N_{mt} - \int_{t_0}^{t}\lambda_m\left(s\right)ds$ is a martingale under Q.

Furthermore, $\lambda_m\left(t\right) = -\vartheta_{m+K}\left(t,\bullet\right)$ (equation 6.17)

Proof: From Girsanov's theorem for marked point processes applied to equation 6.6. •

We assume that $\lambda_m\left(t\right)$, for each m, is a deterministic function of at most t and that it is strictly positive and bounded, which is therefore (from equation 6.17), also an an assumption that $\vartheta_{m+K}\left(t,\bullet\right)$, for each m, is a deterministic function of at most t ie $\vartheta_{m+K}(t,\bullet)=\vartheta_{m+K}(t)$ and that it is strictly negative and bounded.

Proposition 6.5 : The price $D(t)$ of the derivative, at time t, satisfies the following partial integro-differential equation:

$$
\begin{aligned}
0 &= D_t dt + \frac{1}{2}\sum_{i=1}^{K+M+1}\sum_{j=1}^{K+M+1} D_{H_iH_j} dH_i dH_j + \sum_{m=1}^{M}\lambda_m(t)\left(D\left(\overline{g}_m\overline{H}+\overline{H}\right)-D(t)\right)dt \\
&-\sum_{i=1}^{K+M+1} D_{H_i}\left(\sum_{m=1}^{M}\left(\lambda_m(t)\,g_m(t,T_i)\right)\right)H_i dt \\
&+\frac{1}{2}D_{PP}dPdP+\sum_{j=1}^{K+M+1} D_{PH_j}dPdH_j
\end{aligned}
\qquad \text{(equation 6.18)}
$$

Proof: From substituting equation 6.17 into equation 6.13. •

Remark 6.6 : This is a partial integro-differential equation which provided we have boundary conditions, in principle, we can solve.

Remark 6.7 : Note that, unlike equation 6.13, equation 6.18 does not involve $\vartheta_{m+K}(t,\bullet)$ for any m.

6.2 Spot jump amplitudes are discrete random variables

Suppose now that the spot jump amplitudes γ_{mt}, for each m, $m=1,\dots,M$, are each discrete random variables, independent and identically distributed, with a finite (in practise, small) number U_m of possible states, ($U_m \geq 1$). That is suppose, for each m,

$\gamma_{mt}=u_{m,i_m}$ with probability v_{i_m} where v_{i_m} are constants such that $v_{i_m}>0$ and $\sum_{i_m=1}^{U_m} v_{i_m}=1$ and where u_{m,i_m} are known constants.

It is easy to see (with the aid of, for example, Karlin and Taylor (1975)) that the number of occurrences of jumps of size u_{m,i_m} in a given time interval has a Poisson distribution with intensity $v_{i_m}\lambda_m(t)$.

Hence the distribution of futures prices is the same as if there were M' Poisson processes (instead of M) with intensities

$$v_{1_1}\lambda_1(t),v_{2_1}\lambda_1(t),\dots,v_{U_1}\lambda_1(t),v_{1_2}\lambda_2(t),v_{2_2}\lambda_2(t),\dots,v_{U_2}\lambda_2(t),\dots,v_{U_M}\lambda_M(t)$$

where $M'=\sum_{m=1}^{M}U_m$, and where each of the M' Poisson processes drives a known constant spot jump amplitude of the form u_{m,i_m}.

Now, of course, the above argument is very intuitive but we have not specified whether the probabilities v_{i_m} are specified under Q^w or Q. The probabilities will be different under the two different probability measures but we assume that the probabilities are constants of the above form but with (in general) different values under Q^w or Q.

If the number of futures contracts available were less than $K + M^{'} + 1$ then the market would be incomplete. However, we will assume that the number of futures contracts available is greater than or equal to $K + M^{'} + 1$. Hence the analysis in section 6.1 is still valid with M replaced by $M^{'}$ because the market is complete. Hence we can use all the results in sections 2 to 5 when the spot jump amplitudes are constant (assumption 2.1) with the number of Poisson processes expanded to $M^{'}$.

6.3 Spot jump amplitudes are continuous random variables

Now suppose that the spot jump amplitudes γ_{mt}, for each m, $m = 1,...,M$ are continuous random variables which are independent and identically distributed and defined on $\left(\Omega, \mathrm{F}, Q^w\right)$.

It is fairly clear that our market will not be complete and therefore there will not be a unique equivalent martingale measure. However, in view of the results of section 6.1, it would be nice to write down the dynamics of the futures commodity prices under Q^w in the form:

$$\frac{dH(t,T_i)}{H(t,T_i)} = \mu_i(t,\bullet)dt + \sum_{k=1}^{K}\sigma_{Hk}(t,T_i)dw_{Hk}(t) - \sigma_P(t,T_i)dw_P(t)$$
$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\phi_m(t,T_i)\right)-1\right)dN_{mt} \qquad \text{(equation 6.19)}$$

where, for each m, γ_{mt} are the spot jump amplitudes, whose distribution is defined under Q^w (with the filtration suitably augmented) and which are independent and identically distributed.

As before, $\lambda_m^w(t)$ is the intensity rate under Q^w (assumed a deterministic function of at most t) and define E_{Nmt}^w to be the expectation operator defined under Q^w conditional on a jump in N_{mt}, for each m.

Then we would like to establish that there exists an equivalent martingale measure Q under which the dynamics of futures commodity prices are martingales of the form:

$$\frac{dH(t,T_i)}{H(t,T_i)} = \sum_{k=1}^{K}\sigma_{Hk}(t,T_i)dz_{Hk}(t) - \sigma_P(t,T_i)dz_P(t)$$
$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\phi_m(t,T_i)\right)-1\right)dN_{mt} - \sum_{m=1}^{M}\lambda_m(t)E_{Nmt}\left(\exp\left(\gamma_{mt}\phi_m(t,T_i)\right)-1\right)dt$$

where the distribution of γ_{mt} is defined under Q, $\lambda_m(t)$ is the intensity rate under Q (assumed a deterministic function of at most t) and E_{Nmt} is the expectation operator defined under Q conditional on a jump in N_{mt} and further that for each m, $\phi_m(t,T_i)$ is as in equation 4.1 and the Brownian motions under Q are defined as in equations 6.15 and 6.16.

When the spot jump amplitudes are continuous random variables, then the term $\left(\exp\left(\gamma_{mt}\phi_m(t,T_i)\right)-1\right)dN_{mt}$, for each m, is a non-linear risk. If we attempt to use the methodology leading to equation 6.4 we see that there is no self-financing portfolio solely composed of futures contracts which can be made risk-free and therefore we must use a different technique.

We will use Girsanov's theorem for marked point processes:

Define L_t via $L_{t_0} = 1$ and

$$\frac{dL_t}{L_t} = -\left(\sum_{k=1}^{K} \vartheta_k(t,\bullet)\, dw_{Hk}(t) - \vartheta_{K+M+1}(t,\bullet)\, dw_P(t)\right)$$
$$+\sum_{m=1}^{M}(\Theta_{mt}-1)\, dN_{mt} - \sum_{m=1}^{M}\lambda_m(t)\, E_{Nmt}^{w}(\Theta_{mt}-1)\, dN_{mt} \qquad \text{(equation 6.20)}$$

where Θ_{mt}, for each m, is a F predictable non-negative process such that $E_{Nmt}^{w}(\Theta_{mt})$ is finite, for all $t \in [t_0, T_{\max}]$ and $\vartheta_k(t,\bullet)$, for each k, $k = 1,2,...,K$ and $\vartheta_{K+M+1}(t,\bullet)$ are F predictable processes.

Define, for each m, $\overline{\Theta}_{mt} \equiv E_{Nmt}^{w}(\Theta_{mt})$. (equation 6.21)

Under technical regularity conditions, L_t is a martingale which we can use as the Radon-Nikodym derivative in a change of measure, from Q^w to Q, via $\dfrac{dQ}{dQ^w} = L_t$.

Define $z_P(t)$ and $z_{Hk}(t)$, for each k, $k = 1,2,...,K$, (as in equations 6.15 and 6.16) which are Brownian motions under Q. Now N_{mt}, for each m, $m = 1,...,M$, are Poisson processes with Q intensity rates $\lambda_m(t)$, that is,

for each m, $m = 1,...,M$, $N_{mt} - \int_{t_0}^{t} \lambda_m(s)\, ds$ is a martingale under Q. We assume that $\lambda_m(t)$, for each m, is a deterministic function of at most t and that it is strictly positive and bounded.

Under any probability measure Q, equivalent to Q^w, the dynamics of $H(t,T_i)$ must satisfy

$$\frac{dH(t,T_i)}{H(t,T_i)} = \mu_i(t,\bullet)\, dt + \sum_{k=1}^{K}\sigma_{Hk}(t,T_i)\, dz_{Hk}(t) - \sigma_P(t,T_i)\, dz_P(t)$$
$$-\sum_{k=1}^{K}\vartheta_k(t,\bullet)\,\sigma_{Hk}(t,T_i)\, dt + \vartheta_{K+M+1}(t,\bullet)\,\sigma_P(t,T_i)\, dt$$
$$+\sum_{m=1}^{M}\lambda_m(t)\, E_{Nmt}\left(\frac{\Theta_{mt}}{\overline{\Theta}_{mt}}\left(\exp(\gamma_{mt}\phi_m(t,T_i))-1\right)\right)dt$$
$$+\sum_{m=1}^{M}\left(\exp(\gamma_{mt}\phi_m(t,T_i))-1\right)dN_{mt} - \sum_{m=1}^{M}\lambda_m(t)\, E_{Nmt}\left(\frac{\Theta_{mt}}{\overline{\Theta}_{mt}}\left(\exp(\gamma_{mt}\phi_m(t,T_i))-1\right)\right)dt$$

(equation 6.22)

and furthermore, for each m,

$$\lambda_m(t) = \lambda_m^w(t)\,\overline{\Theta}_{mt}. \qquad \text{(equation 6.23)}$$

Proposition 6.8 : The futures price $H(t,T_i)$ is a martingale under Q if and only if

$$0=\mu_i(t,\bullet)-\sum_{k=1}^{K}\vartheta_k(t,\bullet)\sigma_{Hk}(t,T_i)+\vartheta_{K+M+1}(t,\bullet)\sigma_P(t,T_i)$$
$$+\sum_{m=1}^{M}\lambda_m(t)E_{Nmt}\left(\frac{\Theta_{mt}}{\overline{\Theta}_{mt}}\left(\exp\left(\gamma_{mt}\phi_m(t,T_i)\right)-1\right)\right) \qquad \text{(equation 6.24)}$$

Proof: Set the drift term in equation 6.22 to zero. •

Remark 6.9 : Equation 6.24 must hold for all (arbitrary) futures contracts maturities and Θ_{mt}, for each m, must be such that for all $t\in[t_0,T_{\max}]$, the futures price to an arbitrary maturity T must be a martingale. This is a complex condition on the drift term for futures prices whose full implications are beyond the scope of this paper (but see Bjork et al. (1997)).

However, we know that equation 6.24 must be satisfied if we are to be sure our model is consistent with no arbitrage. Two circumstances in which it can be satisfied are:

Either
1./ When the spot jump amplitudes γ_{mt} are constant. This corresponds to assumption 2.1 of section 2. Of course, this is the case already considered in section 6.1 (and corresponds to Θ_{mt} being a deterministic function of at most t).

Or
2./ When $\phi_m(t,T_i)\equiv 1$, for all m and for all T_i. This of course means the jump decay coefficient functions are identically equal to zero ie $b_m(t)\equiv 0$ for all t. This corresponds to assumption 2.2 of section 2. In this case, jumps cause parallel shifts in the log of the futures prices across all tenors.

Let us consider the second case in more depth.

Of course, in this case, it is possible to utilise the arguments leading to equation 6.4 by creating a risk-free self-financing portfolio solely from futures contracts. Of course, the market is incomplete because it is not possible to hedge derivatives in general since in general derivatives are non-linear instruments.

The change of measure from Q^w to Q will change the distribution of the spot jump amplitudes γ_{mt}. Girsanov's theorem tells us that the density function of $\exp(\gamma_{mt})$ under Q is:

$$\frac{\Theta_{mt}}{\overline{\Theta}_{mt}}\exp(\gamma_{mt}).$$

We can[5] call the parameters of the distribution under Q risk-neutral parameters. With a slight abuse of notation, and using the density function under Q, we can write the dynamics of futures prices under Q as the following martingale:

---

[5] We note that the change of measure may change the type of distribution of γ_{mt} as well as the parameters although in the case we consider in section 5 where the spot jump amplitudes γ_{mt} are normally distributed, they will (conveniently) be normally distributed under both Q^w and Q if Θ_{mt} is log-normally distributed.

$$\frac{dH(t,T_i)}{H(t,T_i)} = \sum_{k=1}^{K} \sigma_{Hk}(t,T_i)\, dz_{Hk}(t) - \sigma_P(t,T_i)\, dz_P(t)$$
$$+\sum_{m=1}^{M}\left(\exp(\gamma_{mt})-1\right) dN_{mt} - \sum_{m=1}^{M} \lambda_m(t)\, E_{Nmt}\left(\exp(\gamma_{mt})-1\right) dt \qquad \text{(equation 6.25)}$$

Since the market is incomplete, it is not possible to form a risk-free portfolio using futures contracts and bonds to hedge or replicate non-linear derivatives. However it is still possible (analogously to equations 6.9 and 6.10) to create a self-financing portfolio, with zero net aggregate investment, whose expected value (with respect to an equivalent martingale measure) at any future time is zero. Hence, by analogy to equation 6.18, we can show that the price $D(t)$ of any derivative, at time t, satisfies:

$$0 = D_t dt + \frac{1}{2}\sum_{i=1}^{K+M+1}\sum_{j=1}^{K+M+1} D_{H_iH_j} dH_i dH_j + \sum_{m=1}^{M} \lambda_m(t)\, E_{Nmt}\left(D\left(\overline{g}_m \overline{H} + \overline{H}\right) - D(t)\right) dt$$
$$-\sum_{i=1}^{K+M+1} D_{H_i}\left(\sum_{m=1}^{M}\left(\lambda_m(t)\, E_{Nmt}\left(g_m(t,T_i)\right)\right)\right) H_i dt$$
$$+\frac{1}{2} D_{PP} dPdP + \sum_{j=1}^{K+M+1} D_{PH_j} dPdH_j \qquad \text{(equation 6.26)}$$

This is a partial integro-differential equation which, given boundary conditions, in principle, we can solve.

Of course, this does not uniquely determine the price of a derivative since embedded within the Q intensities $\lambda_m(t)$ and the parameters of the distribution, under Q, of the spot jump amplitudes are market prices of risk. The equivalent martingale measure is not unique and corresponding to each possible equivalent martingale measure there will be different intensity rates and different parameters of the distribution of the spot jump amplitudes leading to different derivative prices.

6.4 Spot jump amplitudes are of mixed form

There is of course a further case of interest, beyond those considered in sections 6.1, 6.2 and 6.3. This is of course when, for different m, $m = 1,...,M$, the spot jump amplitudes are of different types. Since we have shown in section 6.2 that the assumption that the spot jump amplitudes are discrete random variables with a finite number of states can be considered as a particular case of when they are constants, we can, without loss of generality, consider the circumstances when some of the spot jump amplitudes are constant and some are continuous random variables.

Without repeating ourselves, it is clear that we can combine the two cases:
The dynamics of futures prices under Q are as in equation 2.10 where it is understood that for each m, $m = 1,...,M$, one of assumptions 2.1 and 2.2 apply.

We can also combine equations 6.18 and 6.26. In addition, we substitute from equations 6.8. Then, provided that for each m, $m = 1,...,M$, one of assumptions 2.1 and 2.2 apply, we know that the price $D(t)$ of any derivative, at time t, satisfies the following partial integro-differential equation:

$$
\begin{aligned}
0 &= \frac{\partial D}{\partial t} + \sum_{m=1}^{M} \lambda_m(t) E_{Nmt}\left(D\left(\bar{g}_m \bar{H} + \bar{H} \right) - D(t) \right) \\
&+ \frac{1}{2} \sum_{i=1}^{K+M+1} \sum_{j=1}^{K+M+1} \frac{\partial^2 D}{\partial H_i \partial H_j} \left\{ \sigma_P(t,T_i)\sigma_P(t,T_j) + \sum_{k=1}^{K}\sum_{l=1}^{K} \rho_{HkHl}\sigma_{Hk}(t,T_i)\sigma_{Hl}(t,T_j) \right\} H(t,T_i) H(t,T_j) \\
&- \frac{1}{2} \sum_{i=1}^{K+M+1} \sum_{j=1}^{K+M+1} \frac{\partial^2 D}{\partial H_i \partial H_j} \left\{ \sum_{l=1}^{K} \rho_{PHl}\sigma_P(t,T_i)\sigma_{Hl}(t,T_j) + \sum_{k=1}^{K} \rho_{PHk}\sigma_P(t,T_j)\sigma_{Hk}(t,T_i) \right\} H(t,T_i) H(t,T_j) \\
&- \sum_{i=1}^{K+M+1} \frac{\partial D}{\partial H_i} \left(\sum_{m=1}^{M} \left(\lambda_m(t) E_{Nmt}\left(g_m(t,T_i) \right) \right) \right) H_i \\
&+ \frac{1}{2} \sigma_P^2(t,T) P^2(t,T) \frac{\partial^2 D}{\partial P^2} \\
&+ \sum_{j=1}^{K+M+1} \frac{\partial^2 D}{\partial P \partial H_j} \left(\sum_{k=1}^{K} \rho_{PHk}\sigma_P(t,T)\sigma_{Hk}(t,T_j) - \sigma_P(t,T)\sigma_P(t,T_j) \right) P(t,T) H(t,T_j)
\end{aligned}
$$

(equation 6.27)

where it is understood that:

$g_m(t,T_i) \equiv \exp\left(\gamma_{mt}\phi_m(t,T_i)\right) - 1$ in the case that for a given m assumption 2.1 is satisfied and

$g_m(t,T_i) \equiv \exp\left(\gamma_{mt}\right) - 1$ in the case that for a given m assumption 2.2 is satisfied.

Of course, if any of the spot jump amplitudes are random variables (or, if all the spot jump amplitudes are constants, the number of futures contracts were less than $K + M + 1$), then the market is incomplete. In section 5, we have shown that standard options have prices of a simple form. We can estimate the parameters of our model, by inverting the market prices of such options (provided there are sufficient options in the market). Embedded within those parameters, specifically the intensity rates and (in the case of assumption 2.2) the parameters of the distribution of the spot jump amplitudes, are market prices of risk which are "fixed by the market" and which therefore also "fix" the equivalent martingale measure Q. This is a standard technique in incomplete markets.

In practise, most commodities markets have futures contracts of many different maturities. For example, there are futures contracts on WTI grade crude oil for more than 120 different maturities. Even for base metals, which are less actively traded than crude oil, the London Metal Exchange trades futures contracts for 27 different maturities on a wide variety of different base metals. Hence, for example, if K were set equal to three, then the number of Poisson processes M could be set to 20 or more and, in the case that all spot jump amplitudes are constants (assumption 2.1), our market would still be complete. In practise, setting M to equal just one or two, say, would be more realistic to allow for an easier calibration whilst still allowing considerable flexibility in the model.

6.5 The dynamics of forward commodity prices

Our aim in this section is to derive the relationship between forward commodity prices and futures commodity prices.

The price of any contingent claim, whose price at time t is $D(t)$, must satisfy the partial integro-differential equation of equation 6.27:

Now let us suppose that the contingent claim matures at time T at which time its payoff is $H(T,T) - k$.

If we seek a solution of equation 6.27 for $D(t)$ in the form

$$D(t)=P(t,T)\left[H(t,T)\exp\left(\xi(t,T)\right)-k\right]$$

for some deterministic function $\xi(t,T)$, which is not dependent upon $P(t,T)$ nor upon $H(t,T)$, then substituting into equation 6.27, many terms cancel and we need

$$0=\frac{\partial\xi(t,T)}{\partial t}\exp\left(\xi(t,T)\right)$$
$$+\left(\sum_{k=1}^{K}\rho_{PHk}\sigma_P(t,T)\sigma_{Hk}(t,T)-\sigma_P^2(t,T)\right)\exp\left(\xi(t,T)\right)$$

Using the boundary condition $D(T)=H(T,T)-k$ implies that $\xi(T,T)=0$ which implies

$$\xi(t,T)=\int_t^T\left(\sum_{k=1}^{K}\rho_{PHk}\sigma_P(u,T)\sigma_{Hk}(u,T)-\sigma_P^2(u,T)\right)du\,.$$

We denote the forward commodity price at time t to (ie for delivery at) time T by $F(t,T)$. From equation 2.6, we know that $F(T,T)=H(T,T)$. Using this and the definition of a forward price, we know the forward price $F(t,T)$ is that value of k which makes $D(t)=0$.
Hence

$$F(t,T)=H(t,T)\exp\left(\int_t^T\left(\sum_{k=1}^{K}\rho_{PHk}\sigma_P(u,T)\sigma_{Hk}(u,T)-\sigma_P^2(u,T)\right)du\right).$$

(equation 6.28)

Using Ito's lemma, the dynamics of the forward commodity price under Q must be:

$$\frac{dF(t,T)}{F(t,T)}=\left(\sigma_P^2(t,T)-\left\{\sum_{k=1}^{K}\rho_{PHk}\sigma_P(t,T)\sigma_{Hk}(t,T)\right\}\right)dt$$
$$+\sum_{k=1}^{K}\sigma_{Hk}(t,T)dz_{Hk}(t)-\sigma_P(t,T)dz_P(t)$$
$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)du\right)\right)-1\right)dN_{mt}$$
$$-\sum_{m=1}^{M}\lambda_m(t)E_{Nmt}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)du\right)\right)-1\right)dt \qquad \text{(equation 6.29)}$$

Remark 6.10 : Note that equation 6.28, expressing the forward commodity price in terms of the futures commodity price, does not depend on $\lambda_m(t)$ or on γ_{mt} for any m. This is intuitive because, of course, it is possible to hedge a forward contract perfectly with a futures contract, whether we make assumption 2.1 or 2.2 for the spot jump amplitudes, since forward contracts (unlike derivatives in general) are linear contracts.

Remark 6.11 : Note that the dynamics of forward commodity prices and those of futures commodity prices differ only by a deterministic drift term. Our model has mostly been expressed in terms of futures commodity prices. However, equations 6.28 and 6.29 show that it would have been straightforward to have worked with forward commodity prices instead.

In analogy to the way that we have defined the instantaneous futures convenience yield forward rate, it is possible (following Miltersen and Schwartz (1998)) to define an instantaneous forward convenience yield forward rate $\delta(t,T)$ at time t to time T via

$$F(t,T)=\frac{C_t}{P(t,T)}\exp\left(-\int_{s=t}^{T}\delta(t,s)ds\right). \qquad \text{(equation 6.30)}$$

However note that, in general, $\delta(t,T)\neq\varepsilon(t,T)$. Indeed, taking logs of equation 6.30, we must have:

$$\int_{s=t}^{T}\left(\varepsilon(t,s)-\delta(t,s)\right)ds=\left(\int_{t}^{T}\left(\left\{\sum_{k=1}^{K}\rho_{PHk}\sigma_P(u,T)\sigma_{Hk}(u,T)\right\}-\sigma_P^2(u,T)\right)du\right)$$

(equation 6.31)

which (as in Miltersen and Schwartz (1998)) shows that $\varepsilon(t,T)$ and $\delta(t,T)$ will coincide if and only if interest-rates (and therefore also bond prices) are deterministic.

One further point worth amplifying is as follows:
Throughout this paper we have assumed that the dynamics of futures commodity prices under Q are of the form:

$$\begin{aligned}\frac{dH(t,T)}{H(t,T)}&=\sum_{k=1}^{K}\sigma_{Hk}(t,T)dz_{Hk}(t)-\sigma_P(t,T)dz_P(t)\\&+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)du\right)\right)-1\right)dN_{mt}\\&-\sum_{m=1}^{M}\lambda_m(t)E_{Nmt}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)du\right)\right)-1\right)dt\end{aligned} \qquad \text{(equation 6.32)}$$

The question might be asked: Why have the term $-\sigma_P(t,T)dz_P(t)$?
In view of our definition of fictitious futures convenience yield bond prices in equations 2.7, 2.8 and 2.9, we have:

$$C_tP_\varepsilon(t,T)=H(t,T)P(t,T)$$

Using Ito's lemma for jump diffusions implies:

$$\begin{aligned}\frac{d\left(H(t,T)P(t,T)\right)}{H(t,T)P(t,T)}&=\frac{d\left(C_tP_\varepsilon(t,T)\right)}{C_tP_\varepsilon(t,T)}=\sum_{k=1}^{K}\sigma_{Hk}(t,T)dz_{Hk}(t)\\&+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)du\right)\right)-1\right)dN_{mt}\\&-\sum_{m=1}^{M}\lambda_m(t)E_{Nmt}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)du\right)\right)-1\right)dt\end{aligned} \qquad \text{(equation 6.33)}$$

Hence the term $-\sigma_P(t,T)\,dz_P(t)$ cancels out. Indeed we have already seen that equations 3.4 and 3.7 do not involve $dz_P(t)$.

If we did not have the term $-\sigma_P(t,T)\,dz_P(t)$ in equation 6.32 and equation 2.10, then the dynamics of the value of the commodity and those of fictitious futures convenience yield bond prices (and hence also those of the instantaneous futures convenience yield forward rates and short rate) would depend on the Brownian motion driving interest-rates and bond prices. Although there would be nothing wrong with this, it just seems less intuitively appealing. Of course, in a sense, writing the dynamics of futures commodity prices in the form of equation 6.32 is a non-assumption in that given the dynamics in equation 6.32 for any K, we can rewrite them in the form

$$\begin{aligned}\frac{dH(t,T)}{H(t,T)} &= \sum_{k=1}^{K'}\sigma_{Hk}(t,T)\,dz_{Hk}(t)\\ &+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du\right)\right)-1\right)dN_{mt}\\ &-\sum_{m=1}^{M}\lambda_m(t)\,E_{Nmt}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du\right)\right)-1\right)dt\end{aligned}\qquad\text{(equation 6.34)}$$

where $K' \equiv K+1$ and $dz_{HK'}(t)\equiv dz_P(t)$ and

$$\sigma_{HK'}(t,T)\equiv-\sigma_P(t,T)=-\frac{\sigma_r}{\alpha_r}+\frac{\sigma_r}{\alpha_r}\exp\left(-\alpha_r(T-t)\right).$$

The volatility term $\sigma_{HK'}(t,T)$ is still of the form of equation 2.15 and hence the model is still Markovian in the same number of state variables as before.

7. Conclusions

We have considered a simple and tractable multi-factor jump-diffusion model for the evolution of futures commodity prices consistent with any initial term structure. We have related this model to the evolution of forward commodity prices and to the value of the commodity. We have shown that the value of the commodity exhibits mean reversion. We have shown that stochastic interest-rates can also contribute to mean reversion in the value of the commodity. Particularly noteworthy is the way in which we have related our model to stochastic convenience yields which themselves exhibit mean reversion and also, depending on the form of the model, may exhibit jumps. We have been able to model stochastic convenience yields without having to make assumptions about their associated market price of risk. Whilst some of the expressions appear quite long, the model described in this paper is conceptually straightforward. The model is highly amenable to Monte Carlo simulation. Furthermore equations have been derived which allow for the simulation of commodity prices without discretisation error. We have shown that the prices of standard options have semi-analytical solutions. This opens the possibility of calibrating the model through deriving implied parameters from the market prices of options.

Table 1 (Example 1) No Poisson processes

$$T_2 = T_1 + 0.125$$

All options are standard European calls on futures. The values of T_1 are down the first column.

Below are the option prices

Strikes ->	75	80	95	110	115
0.25	19.812	15.081	4.213	0.515	0.214
0.5	19.805	15.421	5.530	1.292	0.730
0.75	19.836	15.702	6.367	1.924	1.219
1	19.860	15.920	6.986	2.447	1.652
2	19.869	16.468	8.605	4.023	3.061
3	19.789	16.766	9.656	5.203	4.185

Below are the implied Black (1976) volatilities (expressed as percentages), using a price of 95.

0.25	22.525%
0.5	21.177%
0.75	20.167%
1	19.407%
2	17.789%
3	17.154%

The above option prices are in the pure-diffusion case and are priced using equation 5.14 (reduced to the no-jump case).

In all cases, $H(t,T_2) = 95$ for all T_2 and $P(t,T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

In all cases, we have two Brownian motions (in addition to the Brownian motion driving interest-rates) and

$\eta_{H1} = 0.266$, $\eta_{H2} = 0.249/1.045 \approx 0.23827751196$, $\chi_{H1} = 0.0$, $\chi_{H2} = -0.249/1.045$

$a_{H2} = 1.045$

$\sigma_r = 0.0096$, $\alpha_r = 0.2$

$\rho_{H1H2} = -0.805$, $\rho_{PH1} = -0.0964$, $\rho_{PH2} = 0.1243$

Table 2 (Example 2) One Poisson process

$T_2 = T_1 + 0.125$

All options are standard European calls on futures. The values of T_1 are down the first column.

Below are the option prices

Strikes->	75	80	95	110	115
0.25	19.8460	15.1892	4.7491	0.9345	0.5129
0.5	19.9199	15.6447	6.0987	1.7881	1.1347
0.75	19.9956	15.9661	6.9049	2.4148	1.6419
1	20.0410	16.1943	7.4844	2.9143	2.0654
2	20.0639	16.7238	8.9826	4.3986	3.4127
3	19.9732	16.9906	9.9626	5.5164	4.4828

Below are the standard errors for the option prices above

Strikes->	75	80	95	110	115
0.25	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001
0.5	<0.0001	<0.0001	0.0001	0.0003	0.0004
0.75	0.0001	0.0002	0.0005	0.0008	0.0009
1	0.0003	0.0004	0.0009	0.0014	0.0013
2	0.0009	0.0012	0.0019	0.0025	0.0026
3	0.0011	0.0014	0.0021	0.0028	0.0028

Below are the implied Black (1976) volatilities (expressed as percentages) of the option prices above

Strikes->	75	80	95	110	115
0.25	24.107%	24.371%	25.393%	26.769%	27.258%
0.5	22.734%	22.885%	23.355%	23.882%	24.063%
0.75	21.491%	21.590%	21.874%	22.167%	22.270%
1	20.525%	20.598%	20.798%	20.988%	21.049%
2	18.442%	18.489%	18.575%	18.651%	18.676%
3	17.596%	17.633%	17.702%	17.756%	17.764%

For the above option prices, we have used equation 5.14 with $M = 1$ and

$\lambda_1 = 0.75$, $\beta_1 = 0.22$, $b_1 = 2.0$.

In all cases, $H(t, T_2) = 95$ for all T_2 and $P(t, T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

Table 3 (Example 3) Two Poisson processes

$T_2 = T_1 + 0.125$

All options are standard European calls on futures. The values of T_1 are down the first column.

Below are the option prices

Strikes->	75	80	95	110	115
0.25	20.109	15.693	5.924	1.885	1.279
0.5	20.695	16.817	8.159	3.626	2.744
0.75	21.310	17.769	9.704	5.021	4.008
1	21.867	18.563	10.911	6.188	5.103
2	23.530	20.801	14.208	9.626	8.452
3	24.564	22.187	16.306	11.990	10.831

Below are the implied Black (1976) volatilities (expressed as percentages) of the option prices above

Strikes->	75	80	95	110	115
0.25	31.022%	30.800%	31.685%	34.313%	35.195%
0.5	30.227%	30.373%	31.281%	32.531%	32.947%
0.75	29.858%	30.046%	30.785%	31.622%	31.897%
1	29.588%	29.769%	30.382%	31.020%	31.227%
2	29.015%	29.143%	29.509%	29.847%	29.953%
3	28.802%	28.900%	29.168%	29.403%	29.476%

For the above option prices, we have used equation 5.14 with $M = 2$ and

$\lambda_1 = 0.75$, $\beta_1 = 0.22$, $\upsilon_1 = 0.01$, $b_1 = 0.0$, $\lambda_2 = 0.75$, $\beta_2 = -0.15$, $\upsilon_2 = 0.01$, $b_2 = 0.0$

In all cases, $H(t,T_2) = 95$ for all T_2 and $P(t,T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

Table 4 (Example 4) No Poisson processes

$T_1 = 1$ and $T_2 = 1.125, 2, 3$

All options are standard European calls on futures. The values of T_2 are down the first column.

Below are the option prices

Strikes ->					
	75	80	95	110	115
1.125	19.860	15.920	6.986	2.447	1.652
2	19.432	15.250	5.818	1.554	0.933
3	19.402	15.199	5.720	1.485	0.880

Below are the implied Black (1976) volatilities (expressed as percentages), using a price of 95.

1.125	19.407%
2	16.156%
3	15.883%

The above option prices are in the pure-diffusion case and are priced using equation 5.14 (reduced to the no-jump case).

In all cases, $H(t, T_2) = 95$ for all T_2 and $P(t, T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

In all cases, we have two Brownian motions (in addition to the Brownian motion driving interest-rates) and

$\eta_{H1} = 0.266$, $\eta_{H2} = 0.249/1.045 \approx 0.23827751196$, $\chi_{H1} = 0.0$, $\chi_{H2} = -0.249/1.045$
$a_{H2} = 1.045$
$\sigma_r = 0.0096$, $\alpha_r = 0.2$
$\rho_{H1H2} = -0.805$, $\rho_{PH1} = -0.0964$, $\rho_{PH2} = 0.1243$

Table 5 (Example 5) One Poisson process

$T_1 = 1$ and $T_2 = 1.125, 2, 3$

All options are standard European calls on futures. The values of T_2 are down the first column.

Below are the option prices

	Strikes->				
	75	80	95	110	115
1.125	20.0410	16.1943	7.4844	2.9143	2.0654
2	19.4375	15.2592	5.8365	1.5680	0.9434
3	19.4020	15.1988	5.7202	1.4853	0.8801

Below are the standard errors for the option prices above

	Strikes->				
	75	80	95	110	115
1.125	0.0003	0.0004	0.0009	0.0014	0.0013
2	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001
3	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001

Below are the implied Black (1976) volatilities (expressed as percentages) of the option prices above, using a price of 95.

	Strikes->				
	75	80	95	110	115
1.125	20.525%	20.598%	20.798%	20.988%	21.049%
2	16.162%	16.186%	16.207%	16.213%	16.215%
3	15.830%	15.860%	15.884%	15.889%	15.889%

For the above option prices, we have used equation 5.14 with $M = 1$ and

$\lambda_1 = 0.75$, $\beta_1 = 0.22$, $b_1 = 2.0$

In all cases, $H(t, T_2) = 95$ for all T_2 and $P(t, T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

Table 6 (Example 6) Two Poisson processes

$T_1 = 1$ and $T_2 = 1.125, 2, 3$

All options are standard European calls on futures. The values of T_2 are down the first column.

Below are the option prices

Strikes ->	75	80	95	110	115
1.125	21.867	18.563	10.911	6.188	5.103
2	21.379	17.976	10.198	5.560	4.526
3	21.341	17.929	10.141	5.512	4.482

Below are the implied Black (1976) volatilities (expressed as percentages) of the option prices above, using a price of 95.

Strikes ->	75	80	95	110	115
1.125	29.588%	29.769%	30.382%	31.020%	31.227%
2	27.416%	27.624%	28.382%	29.178%	29.431%
3	27.239%	27.451%	28.223%	29.034%	29.292%

For the above option prices, we have used equation 5.14 with $M = 2$ and

$$\lambda_1 = 0.75,\ \beta_1 = 0.22,\ \upsilon_1 = 0.01,\ b_1 = 0.0$$
$$\lambda_2 = 0.75,\ \beta_2 = -0.15,\ \upsilon_2 = 0.01,\ b_2 = 0.0$$

In all cases, $H(t,T_2) = 95$ for all T_2 and $P(t,T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

Table 7 (Example 7) One Poisson process

$T_1 = 1$ and $T_2 = 1.125, 2, 3$

All options are standard European calls on futures. The values of T_2 are down the first column.

Below are the option prices

Strikes ->	75	80	95	110	115
1.125	21.103	17.694	9.983	5.433	4.422
2	20.555	17.029	9.213	4.798	3.850
3	20.511	16.976	9.152	4.750	3.807

Below are the implied Black (1976) volatilities (expressed as percentages) of the option prices above, using a price of 95.

Strikes ->	75	80	95	110	115
1.125	26.129%	26.572%	27.779%	28.801%	29.105%
2	23.410%	24.020%	25.626%	26.903%	27.272%
3	23.178%	23.808%	25.456%	26.756%	27.131%

For the above option prices, we have used equation 5.14 with $M = 1$ and

$\lambda_1 = 0.75$, $\beta_1 = 0.22$, $b_1 = 0.0$

In all cases, $H(t, T_2) = 95$ for all T_2 and $P(t, T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

Table 8 (Example 8) One Poisson process

$T_1 = 1$ and $T_2 = 1.125, 2, 3$

All options are standard European calls on futures. The values of T_2 are down the first column.

Below are the option prices

	Strikes->				
	75	80	95	110	115
1.125	19.9167	16.0069	7.1419	2.5886	1.7760
2	19.4323	15.2502	5.8184	1.5546	0.9330
3	19.4019	15.1986	5.7199	1.4850	0.8799

Below are the standard errors for the option prices above

	Strikes->				
	75	80	95	110	115
1.125	0.0003	0.0004	0.0007	0.0008	0.0007
2	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001
3	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001

Below are the implied Black (1976) volatilities (expressed as percentages) of the option prices above, using a price of 95.

	Strikes->				
	75	80	95	110	115
1.125	19.748%	19.780%	19.843%	19.896%	19.916%
2	16.114%	16.138%	16.157%	16.161%	16.162%
3	15.829%	15.859%	15.883%	15.888%	15.888%

For the above option prices, we have used equation 5.14 with $M = 1$ and

$\lambda_1 = 0.75$, $\beta_1 = 0.22$, $b_1 = 4.0$.

In all cases, $H(t, T_2) = 95$ for all T_2 and $P(t, T_1) = \exp(-0.05(T_1 - t))$ for all T_1.

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