Optimal Hedging of Variance Derivatives

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- We are indebted to Roger Lee for spotting an error in the first version of the paper on which this talk is based.
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- A number of investment banks are reputed to have lost significant amounts of money on their variance derivatives books in Autumn 2008 as stock indices moved by 7 % or more in a day.
- This makes very pertinent the question of how to optimally hedge variance derivatives.
- Nearly all papers on variance swaps have focussed on the log-contract replication approach (eg. Neuberger (1990), (1994), (1996), Dupire (1993), Demeterfi et al. (1999), Carr and Lee (2008)).
- This approach works by noting that, under the assumption that the stock price process has continuous sample paths, the payoff of a (continuously monitored) variance swap can be perfectly hedged by a static position of being long 2 log-forward-contracts and by a dynamic position of being short $2/F(t,T)$ units of forward contracts on the stock, where $F(t,T) \equiv F(t)$ is the forward stock price, at time t, to time T.
- We will henceforth refer to this approach as the "standard $2+2$ log-contract replication" approach.

- Proof:

$$\frac{dF(t)}{F(t)} = \sigma(F(t), t, \bullet)dz_t \;\Rightarrow d\log F(t) = -\frac{1}{2}\sigma^2(F(t), t, \bullet)dt + \sigma(F(t), t, \bullet)dz_t.$$

Hence,

$$2\,\frac{dF(u)}{F(u)} - 2\,d\log F(u) = \sigma^2(F(u), u, \bullet)du.$$

Now integrate from time $t_0 \equiv 0$ to maturity T:

$$\begin{aligned}\int_{t_0}^{T} \sigma^2(F(u), u, \bullet)du &= \int_{t_0}^{T} \Big(2\,\frac{dF(u)}{F(u)} - 2\,d\log F(u)\Big) \\ &= \Big(\int_{t_0}^{T} 2\,\frac{dF(u)}{F(u)}\Big) - 2\Big(\log F(T) - \log F(t_0)\Big).\end{aligned}$$

- In the assumed absence of arbitrage, this also yields the price of a continuously monitored variance swap.
- In words, the price of a continuously monitored variance swap equals (minus) two times the price of a log-forward-contract.

- Main assumption: Continuous sample paths i.e. no jumps.
- But every empirical study (even before Autumn 2008) shows that stocks and stock indices exhibit jumps in their dynamics and that jumps are necessary to fit implied vol. surfaces, etc.
- The "standard $2+2$ log-contract replication" approach is often described as model-independent (which is true in some ways eg it works with local vol., stochastic vol, a mixture of the two - or to put it another way, it works when the log of the stock price is Brownian motion time-changed by essentially any continuous time-change process), but actually it assumes away that which is empirically most important (i.e. jumps).
- Can we do better? This is the subject of my talk.
- Actually, we can do **much** better - and our results have a considerable degree of robustness to model (mis-)specification.

- We define the initial time (today) by $t_0 \equiv 0$ and denote calendar time by t, $t \geq t_0$. Consider a market, which we assume to be free of arbitrage. There is a stock whose forward price, at time t, to time T, is $F(t,T)$. We assume that interest-rates and dividend yields are deterministic and finite.
- The absence of arbitrage guarantees the existence of a risk-neutral equivalent martingale measure. However, as we will utilise Lévy processes, the market is incomplete and, hence, the risk-neutral equivalent martingale measure is not unique. We will assume that one such measure $\mathbb{Q}$ has been fixed on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{Q})$. We denote by $\mathbb{E}_t^{\mathbb{Q}}[]$ the conditional expectation, under $\mathbb{Q}$, at time t.
- We construct the stock price process by assuming that the log of the stock price is a time-change of (possibly, multiple) Lévy processes.

- We have a Lévy process (eg Brownian motion, Merton (1976) or Kou (2002) jump-diffusion, Variance Gamma or CGMY) denoted by X_t, satisfying $X_{t_0} = 0$. We mean-correct X_t so that $\exp(X_t)$ is a (non-constant) martingale with respect to the natural filtration generated by X_t i.e. that $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(X_t)] = \exp(X_{t_0}) = 1$ for all $t \geq t_0$.
- Lévy-Khinchin formula implies we can write the (mean-corrected) characteristic exponent $\overline{\psi}_X(z)$ in the form:

$$-\overline{\psi}_X(z) = -\frac{1}{2}\sigma^2(z^2 + iz) + \int_{-\infty}^{\infty} (\exp(izx) - 1 - iz(\exp(x) - 1))\nu(dx).$$

For future reference, $'$ denotes differentiation i.e. $\overline{\psi}'_X(z) \equiv \partial\overline{\psi}_X(z)/\partial z$, $\overline{\psi}''_X(z) \equiv \partial^2\overline{\psi}_X(z)/\partial z^2$, and further, for $n \geq 3$, $\overline{\psi}^{(n)}_X(z) \equiv \partial^n\overline{\psi}_X(z)/\partial z^n$.

- We assume that we have a (possibly, deterministic) non-decreasing, continuous time-change process denoted by Y_t. We normalise so that $Y_{t_0} = t_0 \equiv 0$.
- In general, Y_t may be correlated with X_t.
- Our assumption, for example, allows Y_t to be of the form $Y_t = \int_{t_0}^{t} y_s ds$ where the activity rate y_t (which must be non-negative) follows, for example, a Heston (1993) square-root process, a non-Gaussian OU process (Barndorff-Nielsen and Shephard (2001)) or it could follow the Heston (1993) plus jumps process of Duffie et al. (2000). In the latter two cases, y_t is discontinuous but Y_t is always continuous.
- (The time-change will allow us to model stochastic volatility / leverage / volatility clustering type effects).

- We time-change the Lévy process X_t by Y_t to get a process X_{Y_t}, with $X_{Y_{t_0}} = 0$.
- The forward stock price $F(t,T)$, at time t, to time T, is assumed to have the following dynamics:

$$F(t,T) = F(t_0,T)\exp(X_{Y_t}).$$

  Note that $F(t,T)$ is a martingale, under $\mathbb{Q}$, in the enlarged filtration generated by $\{X_t \cup Y_t\}$.
- We have already seen that with continuous sample paths, the price of a continuously monitored variance swap equals minus two times the price of a log-forward-contract.

- In general, i.e. with jumps, the price $VS(t_0, T)$, at time t_0, of (the floating leg of) a continuously monitored variance swap maturing at time T equals $-Q_X$ times the price $LFC(t_0, T)$, at time t_0, of a log-forward-contract paying $\log(F(T,T)/F(t_0,T))$ at time T,

where
$$-Q_X \equiv \frac{VS(t_0,T)}{LFC(t_0,T)} = \frac{\overline{\psi}''_X(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}]}{i\overline{\psi}'_X(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}]}$$
$$= \frac{\overline{\psi}''_X(0)}{i\overline{\psi}'_X(0)}.$$
Note that $Q_X > 0$, since $i\overline{\psi}'_X(0) < 0$.

Proof: Carr and Lee (2009), Crosby et al. (2010).

- In particular, there is no up-front cost of entering into a position of being long the floating leg of one variance swap and being long Q_X log-forward-contracts.

- Note that Q_X does not depend on Y_t in any way whatsoever. This means that given the price of a log-forward-contract (which can be replicated from vanilla options), we can price a continuously monitored variance swap without any knowledge of the time-change process Y_t. This gives considerable robustness to model (mis-)specification.

- Basic idea: We construct a self-financing trading strategy as follows: We commence the strategy at time $t_0 \equiv 0$. At each time $t \in [t_0, T]$, we hold a long position in one variance swap and in Θ_t^{LFC} log-forward-contracts. Additionally, we trade dynamically in the underlying stock. Specifically, for all $t \in [t_0, T]$, we hold a short position in $\Delta_t \equiv \phi_t / F(t-, T)$ units of forward contracts on the stock.
- We compute the variance, under $\mathbb{Q}$, of the time T P+L (profit-and-loss) of the self-financing trading strategy i.e. the variance of the residual hedging error.
- It is a non-negative quadratic function of Θ_t^{LFC} and ϕ_t. Minimise by differentiating w.r.t. portfolio weight and setting the resulting equation to zero.
- Can do this analytically (does not need Monte Carlo - see paper for full details)

- For simplicity, I'll just write the equations with deterministic time-changes.
- The P+L of the self-financing strategy, at time T, is:

$$
\begin{aligned}
\epsilon(T) \;\equiv\; & \int_{t_0}^{T} y_u \int_{-\infty}^{\infty} x^2 \big(\mu(dx) - \nu(dx)\big) du \\
+ \; & \int_{t_0}^{T} \Theta_u^{\mathrm{LFC}} y_u \Big(\sigma dW_u + \int_{-\infty}^{\infty} x \big(\mu(dx) - \nu(dx)\big) du\Big) \\
- \; & \int_{t_0}^{T} \Delta_u F(u-, T) y_u \Big(\sigma dW_u + \int_{-\infty}^{\infty} (\exp(x) - 1) \big(\mu(dx) - \nu(dx)\big) du\Big) \\
= \; & \int_{t_0}^{T} y_u \int_{-\infty}^{\infty} \Big(x^2 + \Theta_u^{\mathrm{LFC}} x - \phi_u (\exp(x) - 1)\Big) \big(\mu(dx) - \nu(dx)\big) du \\
+ \; & \int_{t_0}^{T} \Big(\Theta_u^{\mathrm{LFC}} - \phi_u\Big) y_u \sigma dW_u, \text{ using } \phi_t \equiv \Delta_t F(t-, T).
\end{aligned}
$$

- From Ito's isometry formula, the variance, under $\mathbb{Q}$, of the time T P+L of the self-financing strategy is:

$$
\begin{aligned}
Var^{\mathbb{Q}}_{t_0}[\epsilon(T)] &\equiv \mathbb{E}^{\mathbb{Q}}_{t_0}[\Big(\int_{t_0}^{T} y_u(\Theta^{\mathrm{LFC}}_u - \phi_u)^2\sigma^2 du\Big) \\
&+ \Big(\int_{t_0}^{T} y_u \int_{-\infty}^{\infty} \big(x^2 + \Theta^{\mathrm{LFC}}_u x - (\phi_u(\exp(x)-1))\big)^2 \nu(dx)du\Big)] \\
&= \mathbb{E}^{\mathbb{Q}}_{t_0}[\int_{t_0}^{T} y_u\Big(\phi_u^2(-\overline{\psi}_X(-2i)) \\
&- 2\phi_u\Big(\Theta^{\mathrm{LFC}}_u i(\overline{\psi}'_X(-i) - \overline{\psi}'_X(0)) + (\overline{\psi}''_X(-i) - \overline{\psi}''_X(0))\Big) \\
&- \overline{\psi}^{(4)}_X(0) - 2\Theta^{\mathrm{LFC}}_u i\overline{\psi}^{(3)}_X(0) + \Theta^{\mathrm{LFC}\,2}_u \overline{\psi}''_X(0)\Big)du].
\end{aligned}
$$

This is a non-negative quadratic function of ϕ_u and Θ^{LFC}_u. We minimise by differentiating with respect to ϕ_u and Θ^{LFC}_u and setting to zero.

- We hedge a (static) long position in one variance swap.
- We consider two types of hedging strategy labelled A and B.
- The first type of hedging strategy (labelled hedging strategy A) consists of a static position in $\Theta_t^{\mathrm{LFC}} = Q_X$ log-forward-contracts and a dynamic short position in $\Delta_t \equiv \phi_t / F(t-, T)$ forward contracts on the underlying stock. The static position Q_X is motivated by slide "Important result" (but is not necessarily optimal).
- The second type of hedging strategy (labelled hedging strategy B) consists of a, possibly, dynamic position in Θ_t^{LFC} log-forward-contracts and a dynamic short position in $\Delta_t \equiv \phi_t / F(t-, T)$ forward contracts on the underlying stock.

- For hedging strategy A, we have $\Theta_t^{\mathrm{LFC}} = Q_X$ (by design) and we find that the optimal value $\hat{\Delta}_t$ which minimises the variance is:

$$\hat{\Delta}_t \equiv \frac{\hat{\phi}_t}{F(t-,T)}, \text{ where}$$
$$\hat{\phi}_t = \frac{(\overline{\psi}''_X(-i) - \overline{\psi}''_X(0)) + Q_X\, i(\overline{\psi}'_X(-i) - \overline{\psi}'_X(0))}{-\overline{\psi}_X(-2i)}.$$

- Sanity check: For Brownian motion with volatility σ, $\overline{\psi}_X(z) = \frac{1}{2}\sigma^2(z^2 + iz)$, $i\overline{\psi}'_X(0) = -\frac{1}{2}\sigma^2$, $\overline{\psi}''_X(0) = \sigma^2$. Implies: $Q_X = -\frac{\sigma^2}{-\frac{1}{2}\sigma^2} = 2$.
- Further, $\overline{\psi}_X(-2i) = -\sigma^2$, which implies: $\hat{\phi}_t = 2$.
- This agrees with standard results i.e. the standard $2+2$ log-contract replication approach naturally appears as a special case of our analysis. Further, for this special case, substituting back in, the variance $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ is identically equal to zero.
- $\Rightarrow$ perfect hedge.

- For hedging strategy A, we also consider the special case of a compound Poisson process with a fixed jump amplitude a_1 (with no diffusion component). Substituting in the relevant characteristic function, we find:
- $Q_X = a_1^2/(\exp(a_1) - 1 - a_1)$.
- $\hat{\phi}_t = a_1^2/(\exp(a_1) - 1 - a_1)$.
- Further, for this special case, substituting back in, the variance is identically equal to zero.
- $\Rightarrow$ perfect hedge.
- In the limit that $a_1 \rightarrow 0$, we find:

$$\hat{\phi}_t = Q_X = \frac{a_1^2}{(\exp(a_1) - 1 - a_1)} \approx \frac{2}{(1 + (a_1/3))}.$$

We see that when a_1 is small but positive, $\hat{\phi}_t = Q_X$ is just below two and when a_1 is small but negative, $\hat{\phi}_t = Q_X$ is just above two. In either case, as $a_1 \rightarrow 0$, $\phi_t \rightarrow 2$, which is the same as the case of Brownian motion.

- For hedging strategy B, we optimise over Θ_t^{LFC} (the position in log-forward-contracts) and over ϕ_t (recall $\Delta_t \equiv \phi_t / F(t-, T)$ is the position in forward contracts on the underlying stock).
- It turns out that, (at least in the model set-up I have given you today (see the paper for a generalisation)), the position in log-forward-contracts is constant in time i.e. it is a static buy-and-hold position (which is important as dynamic positions would incur significant transactions costs).
- Further, in this special case, ϕ_t and Θ_t^{LFC} do not depend upon the time-change process in any way $\Rightarrow$ considerable degree of robustness to model (mis-)specification.

- We now consider some numerical examples which compare three possible hedging strategies.
- The first hedging strategy is the standard $2 + 2$ log-contract replication approach (sets $\phi_t = 2$, $\Theta_t^{\text{LFC}} = 2$).
- The second and third are hedging strategies A and B respectively which we described earlier.
- We stress again that the values of ϕ_t, Θ_t^{LFC} are constant (to repeat, this has the additional benefit of robustness to transactions costs).
- We consider the hedging of a long position in one variance swap with maturity $T = 0.5$.
- We consider three sets of numerical results - each with six different sets of process parameters. The first uses combinations of a Brownian motion and upto three compound Poisson processes with fixed jump amplitudes together with a deterministic time-change. The second uses CGMY processes with a deterministic time-change. The third uses stochastically time-changed CGMY processes (there are more results in the paper).

- Table 1.
  We consider six combinations (labelled params 1 to params 6) of a Brownian motion and upto three compound Poisson processes, with intensity rates λ_1, λ_2 and λ_3 and with fixed jump amplitudes a_1, a_2 and a_3. We assume a deterministic time-change (not necessarily of the form $Y_t \equiv t$).

	λ_1	a_1	λ_2	a_2	λ_3	a_3	Vol	Skewness swap price	Q_X
params 1	1.00000000	-0.2	0	0	0	0	0.15	-0.00400	2.0846708
params 2	1.53186275	-0.2	0.76593137	0.04	0	0	0	-0.00610	2.1320914
params 3	0.98039216	-0.2	0.49019608	0.04	0	0	0.15	-0.00391	2.0825752
params 4	1.50240385	-0.2	0.75120192	0.04	0.75120192	-0.04	0	-0.00601	2.1299626
params 5	0.96153846	-0.2	0.48076923	0.04	0.48076923	-0.04	0.15	-0.00385	2.0812748
params 6	0.54086538	-0.2	0.27043269	0.04	0.27043269	-0.04	0.2	-0.00216	2.0449185

  For all parameters, the (annualised) variance swap rate expressed as a volatility is 0.25.

	params 1	params 2	params 3	params 4	params 5	params 6
$2+2$						
ϕ_t	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**3.2219755**	**4.9358021**	**3.1589133**	**4.8410503**	**3.0982722**	**1.7427781**
Hedge strategy A						
$\hat{\phi}_t$	2.0815517	2.1316674	2.0793692	2.1293857	2.0780750	2.0420725
Θ_t^{LFC}	2.0846708	2.1320914	2.0825752	2.1299626	2.0812748	2.0449185
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.1843912**	**0.0048679**	**0.1987352**	**0.0080840**	**0.2139058**	**0.5419420**
Hedge strategy B						
$\hat{\phi}_t$	2.1355255	2.1066839	2.1339678	2.1236177	2.1344956	2.1350182
$\hat{\Theta}_t^{\text{LFC}}$	2.1355255	2.1093850	2.1341001	2.1247118	2.1345689	2.1350425
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0**	**0.0**	**0.0040225**	**0.0077286**	**0.0058494**	**0.0033147**

Notice we get perfect hedges in some special cases.

- With combinations of Brownian motions and compound Poisson processes with fixed jump amplitudes, as we increase the number of hedging instruments over which we optimise (underlying, log-forward-contracts), we increase from 1 to 2 the number of underlying stochastic processes that we can perfectly hedge against. This is highly intuitive.
- For example, for hedging strategy B (two instruments, i.e. underlying and log-forward-contracts), we can perfectly hedge when there are two stochastic processes (one Poisson + Brownian motion or two Poisson).

- Table 2.
  We consider six combinations (labelled params 1 to params 6) of a generalised CGMY process with a deterministic time-change.

	C_{Up}	C_{Down}	G	M	Y_{Up}	Y_{Down}	Vol	Skewness swap price	Q_X
params 1	0.60283195	0.04075144	1.64	16.9	-2.9	1.54	0	-0.00876	2.1675629
params 2	0.10998598	0.03170896	0.697	22	-3.65	1.45	0	-0.02466	2.4271496
params 3	0.08888068	0.60125165	3.34	14.64	0.165	0.165	0	-0.01696	2.3517572
params 4	0.60125165	0.08888068	14.64	3.34	0.165	0.165	0	0.01696	1.6245175
params 5	10.8377161	10.8377161	22.56	22.56	0.14	0.14	0	0	1.9982574
params 6	0.82244372	0.82244372	5.64	5.64	0.14	0.14	0	0	1.9719659

	params 1	params 2	params 3	params 4	params 5	params 6
$2+2$						
ϕ_t	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.1003116**	**1.7636830**	**0.1296205**	**0.8684322**	**0.0001355**	**0.0422403**
Hedge strategy A						
$\hat{\phi}_t$	2.1395386	2.3243141	2.3158950	1.5145212	1.9947582	1.9120745
Θ_t^{LFC}	2.1675629	2.4271496	2.3517572	1.6245175	1.9982574	1.9719659
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0747423**	**1.2868488**	**0.0540592**	**0.2835614**	**0.0000966**	**0.0280183**
Hedge strategy B						
$\hat{\phi}_t$	3.0508893	4.4583264	3.1344999	0.9105117	1.9879848	1.8068255
$\hat{\Theta}_t^{\mathrm{LFC}}$	2.9968102	4.1897424	3.0142373	0.7132527	1.9914543	1.8587165
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0286036**	**0.5441049**	**0.0187595**	**0.0614244**	**0.0000962**	**0.0262567**

- When $Q_X \gg 2$ (which implies that the $\mathbb{Q}$ - distribution of stock returns is negatively skewed which is empirically the case for equities), then $\hat{\phi}_t \gg 2$ and $\hat{\Theta}_t^{\mathrm{LFC}} \gg 2$.
- When $Q_X \ll 2$, then $\hat{\phi}_t \ll 2$ and $\hat{\Theta}_t^{\mathrm{LFC}} \ll 2$.
- When jumps are small and symmetric (ie M and G are large and equal eg. params 5) then optimal values $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$ are close to 2 (as also is Q_X).
- Hedging strategy B always outperforms hedging strategy A which, in turn, always outperforms the standard $2 + 2$ log-contract replication approach.

- Table 3.

  We consider six combinations (labelled params 1 to params 6) of a generalised CGMY process time-changed by either a Heston (1993) activity-rate process (params 1 to 5) or a non-Gaussian OU process (the Gamma-OU process of Barndorff-Nielsen and Shephard (2001)) (params 6).

  All parameters were obtained from calibrations to market prices of vanilla options on S & P 500 and are quoted from Schoutens (2003) and from Carr, Geman, Madan and Yor (2003).

	C_{Up}	C_{Down}	G	M	Y_{Up}	Y_{Down}	Vol	Skewness swap price	Q_X
params 1	0.00740000	0.00740000	0.1025	11.394	1.6765	1.6765	0	-0.06977	2.7294158
params 2	0.16350000	0.04713705	0.6965	21.97	-3.65	1.45	0	-0.01272	2.4274086
params 3	0.35870000	0.01886762	0.4231	24.64	-4.51	1.67	0	-0.01419	2.3727413
params 4	0.40410000	0.02731716	1.64	16.91	-2.9	1.54	0	-0.00385	2.1675632
params 5	2.04400000	0.17476200	3.68	52.86	-2.12	1.22	0	-0.01054	2.1349535
params 6	0.04150000	0.04150000	3.9134	30.6322	1.3664	1.3664	0	-0.00182	2.0769284

- Table 3 continued.
  The activity rate for params 1 to params 5 follows a Heston (1993) process of the form:

$$dy_t = \kappa(\eta - y_t)dt + \lambda y_t^{1/2} dz_t, \quad y_{t_0} \equiv y_0, \text{ with } y_0 > 0.$$

	Var swap rate (as vol)	λ	κ	η	y_0	ρ
params 1	0.232270	1.3612	0.3881	1.4012	1	0
params 2	0.179512	0.00022	8.51	0.1497	1	0
params 3	0.190740	0.0006	6.65	0.3469	1	0
params 4	0.165670	2.78E-05	4.85	0.4474	1	0
params 5	0.315297	1.7	15.91	1.3700	1	0
params 6	0.172255	0.8826	$a = 0.5945$	$b = 0.8524$	1	0

	params 1	params 2	params 3	params 4	params 5	params 6
$2+2$						
ϕ_t	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**69.4789464**	**0.9111811**	**1.9158518**	**0.0440515**	**0.0252706**	**0.0032939**
Hedge strategy A						
$\hat{\phi}_t$	2.4383574	2.3244859	2.2640247	2.1395390	2.1218021	2.0679356
Θ_t^{LFC}	2.7294158	2.4274086	2.3727413	2.1675632	2.1349535	2.0769284
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**62.9708207**	**0.6648498**	**1.5885852**	**0.0328228**	**0.0156676**	**0.0024078**
Hedge strategy B						
$\hat{\phi}_t$	10.8956531	4.4599057	5.0370276	3.0508929	2.6186136	2.4716678
$\hat{\Theta}_t^{\text{LFC}}$	9.9700106	4.1910341	4.7686888	2.9968132	2.5907777	2.4615637
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**33.6196207**	**0.2811378**	**0.7166531**	**0.0125611**	**0.0056145**	**0.0010554**

- Very heavily (negatively) skewed Lévy process $\Rightarrow Q_X \gg 2, \Rightarrow \hat{\phi}_t \gg 2$ and $\hat{\Theta}_t^{\text{LFC}} \gg 2$.
- For parameters (params 1) obtained from a calibration to market prices of options on S & P 500, the optimal hedges are **five times** greater than those implicit in the standard $2+2$ log-contract replication approach.
- Hedging strategy B always outperforms hedging strategy A which, in turn, always outperforms the standard $2+2$ log-contract replication approach.
- In the paper, we also consider the use of skewness swaps to help hedge variance swaps (see paper for details).

- The standard $2+2$ log-contract replication approach is very far from optimal in the presence of jumps.
- The good news: We can construct optimal hedges for hedging a long position in one variance swap which are of the form long $\hat{\Theta}_t^{\mathrm{LFC}}$ log-forward-contracts and short $\hat{\phi}_t/F(t-,T)$ forward contracts on the underlying stock.
- The bad news: $\hat{\Theta}_t^{\mathrm{LFC}}$ and $\hat{\phi}_t$ are not 2 (but 2 is the "small jump limit").
- For a wide class of processes (but not always), $\hat{\Theta}_t^{\mathrm{LFC}}$ and $\hat{\phi}_t$ are independent of the time-change ($\Rightarrow$ robust to model (mis-)specification) and constant in time ($\Rightarrow$ robust to transactions costs) but they are highly dependent upon the skew of the Lévy process(es).
- The paper on which this talk is based ("Optimal hedging of variance derivatives") can be found on my website:
  http://www.john-crosby.co.uk .