OPTIMAL HEDGING OF VARIANCE DERIVATIVES

JOHN CROSBY

ABSTRACT. We examine the optimal hedging of derivatives written on realised variance, focussing principally on variance swaps (but, en route, also considering skewness swaps), when the underlying stock price has discontinuous sample paths i.e. jumps. In general, with jumps in the underlying, the market is incomplete and perfect hedging is not possible. We derive easily implementable formulae which give optimal (or nearly optimal) hedges for variance swaps under very general dynamics for the underlying stock which allow for multiple jump processes and stochastic volatility. We illustrate how, for parameters which are realistic for options on the S & P 500 and Nikkei-225 stock indices, our methodology gives significantly better hedges than the standard log-contract replication approach of Neuberger and Dupire which assumes continuous sample paths. Our analysis seeks to emphasize practical implications for financial institutions trading variance derivatives.

Date: First initial draft 16th August 2009. Previous version: 24th February 2011 at 15.38. New version: 14th March 2012 at 19.48.

John Crosby, Grizzly Bear Capital and Centre for Economic and Financial Studies, University of Glasgow and Glasgow University Business School, G12 8RT, United Kingdom.

We thank two anonymous referees as well as an anonymous associate editor for their comments. We also express our gratitude to Chris Adcock (the editor). We are indebted to Roger Lee for spotting a mistake in the first draft of this paper. We are very grateful to Peter Carr and Roger Lee for discussions and for providing us with an early copy of Carr and Lee (2009). We also thank Mark Davis, Bruno Dupire, Jim Gatheral, Dilip Madan, Aleksandar Mijatović, Anthony Neuberger, Jan Obloj and Vimal Raval and seminar and conference participants at Imperial College London, ICBI Global Derivatives, Bachelier Finance Society World Congress, Judge Business School, Cambridge University, Quant Congress and Baruch College, City University of New York for comments. However, any remaining errors are the author's responsibility alone.

Accepted for publication in **The European Journal of Finance**.

Subject classification: Finance; derivatives; incomplete markets; optimal hedging; asset pricing theory.

JEL classification: G12, G13 .

1. Introduction

The market for derivatives written on the variance of the price of an asset such as a stock has grown substantially in recent years and with it the demand for robust and effective ways of hedging such instruments.

Nearly all papers on variance swaps have focussed on the log-contract replication approach (eg. Neuberger (1990), (1994), (1996), Dupire (1993), Demeterfi et al. (1999), Broadie and Jain (2008), Carr and Lee (2010)). In essence, this approach works by noting that, under the assumption that the stock price process has continuous sample paths, the payoff of a (continuously monitored) variance swap can be perfectly hedged by a static position of being long two log-forward-contracts and by a dynamic position of being short $2/S(t)$ units of stock, where $S(t)$ is the stock price at time t. In the assumed absence of arbitrage, this strategy also yields the price of the variance swap. We will henceforth refer to this approach as the "standard $2+2$ log-contract replication" approach.

However, this approach only works when the underlying stock price has continuous sample paths. Even before the extreme events of Autumn 2008, in the wake of the collapse of Lehman Brothers, nearly every empirical study (see Broadie et al. (2007), Carr et al. (2002) and the references therein) had indicated the necessity of incorporating jumps into the dynamics of stocks and stock index futures contracts. This became even more pertinent in Autumn 2008 as stock indices moved by seven per cent or more in a day and the VIX contract (which is essentially a measure of the volatility of the S & P 500 stock index) moved from a pre-crisis level of around 20 per cent to around 70 per cent. Anecdotal stories suggest that a number of investment banks lost significant amounts of money at this time on their variance derivatives books whilst a number of insurance companies are reputed to have lost large sums on their positions in variable annuities (which are effectively complex derivatives with significant exposure to forward volatility).

We will consider the optimal (or nearly optimal) hedging of variance swaps under very general dynamics which allow for multiple jump processes, in the the price of the underlying stock or stock index, as well as Brownian components and also for stochastic volatility (or, more generally, (possibly, multiple) stochastic time-changes). Specifically, the dynamics can be those of (possibly, multiple) time-changed Lévy processes (see Carr et al. (2003)). Although, we will focus on variance swaps, we will, en route, also consider skewness swaps. Our methodology can easily be generalised to other types of variance derivatives such as gamma swaps and self-quantoed variance swaps.

In related work, Carr and Lee (2009) show that, working with time-changed Lévy processes, under stated assumptions, the price of a variance swap is equal to minus Q_X times the price of a log-forward-contract, where Q_X, which they term the "multiplier", depends only on the parameters of the Lévy process(es) and not in any way upon the time-change. They show that Q_X is identically equal to two if the (log of the) underlying stock price process is a (possibly, time-changed) Brownian motion but is some number (in general) different from two if the (log of the) stock price process is discontinuous - specifically, a (possibly, time-changed) Lévy process. In particular, Carr and

Lee (2009) show that Q_X is greater than two for a negatively skewed Lévy process. We note that these results have assumed greater significance since the global financial crisis of Autumn 2008. Before the financial crisis, market prices of variance swaps were close to minus two times the prices of log-forward-contracts (where the latter were inferred from the market prices of co-terminal vanilla options of as many strikes as were available as described in, for example, Demeterfi et al. (1999)) which is in line with (or we might conjecture a self-fulfilling prophecy of) the standard $2+2$ log-contract replication approach. In terms of Carr and Lee (2009), this is saying that before the financial crisis, the "empirical" value (i.e. the value implied by dividing the market prices of variance swaps by (minus) the prices of log-forward-contracts inferred from the market prices of vanilla options) of the multiplier Q_X was approximately two. In the aftermath of the financial crisis, this is no longer the case. Traders have reported to us that, since the financial crisis, for variance swaps written on most major stock indices, the "empirical" value (i.e. the value implied by market prices) of the multiplier Q_X has been consistently in the range 2.10 to 2.25. This is illustrated in Crosby and Davis (2010) (to whom we refer the reader for more details) where they note that the "empirical" value of the multiplier Q_X based on market transactions recorded on 10th December 2010 for variance swaps written on the Nikkei-225 stock index was approximately 2.175 - consistent with a significantly negatively skewed Lévy process (i.e. one with larger down jumps).

Our paper builds upon Carr and Lee (2009) and also a revised and extended version of this paper, Carr et al. (2010). Whilst these papers cover the pricing of variance swaps in detail, only the revised paper discusses hedging and it does so only in the special cases when the Lévy component of the dynamics of the log of the underlying stock price consists of either a Poisson process with a single jump amplitude or consists of two Poisson processes each with a single jump amplitude (which must have opposite sign to one another and with the intensity rates restricted so as to generate piecewise constant sample paths) - both of which are cases when perfect hedging is possible. These specific dynamics might, loosely speaking, be described as "toy" examples to garner intuition. We also illustrate with "toy" examples to enhance intuition and extend the list of those in which perfect hedging is possible. When perfect hedging is possible, the distance from pricing to hedging is small. Therefore, more significantly, we consider much more general stock price dynamics when perfect hedging is not possible.

Compared to Carr et al. (2010), the main new insights and contributions of our paper are as follows:

- 1./ Working with more general dynamics than Carr et al. (2010), we analyse optimal (or nearly optimal) hedges when perfect hedging is not possible. We link these hedges to the characteristic function (and its derivatives) of the underlying Lévy process(es) and therefore to the skewness (and other higher moments) of the Lévy process (which, in turn, relates to the distribution of jumps (and especially their asymmetry) in the underlying stock or stock index).

- 2./ While the optimal (or nearly optimal) hedges that we will derive depend crucially upon the parameters of the Lévy process(es) they do not (under stated assumptions) depend upon the time-change. This is very convenient in that it gives our hedges a degree of robustness to model mis-specification (of the time-change).
- 3./ We show that the standard $2 + 2$ log-contract replication approach naturally appears as the "small jump limit" of our more general analysis.
- 4./ We also (going beyond Carr et al. (2010)) consider the use of skewness swaps to hedge variance swaps (or equivalently, the use of variance swaps to hedge skewness swaps).
- 5./ We also depart from Carr et al. (2010) by providing numerical examples which show both the optimal hedges and measures of the residual error.
- 6./ Traders have reported to us that the use of the standard $2 + 2$ log-contract replication approach is currently universal within investment banks for hedging variance swaps. We show via numerical results, for parameters which are realistic for options on the S & P 500 and Nikkei-225 stock indices, that the standard $2+2$ log-contract replication approach is far from optimal. By contrast, numerical results using our modelling framework demonstrate significantly improved hedging performance.
- 7./ We illustrate how, when the multiplier Q_X is significantly different from two, then the optimal (or nearly optimal) hedges obtained by our methodology depart significantly from the standard $2 + 2$ log-contract replication approach.
- 8./ We also illustrate the relative sizes of the hedging error due to imperfect hedging of the Lévy component(s) of the underlying stock price dynamics and the hedging error due to imperfect hedging of the stochastic time-change(s).
- 9./ We explain why our modelling framework has a degree of robustness to model mis-specification and the possible presence, in practice, of transactions costs and to the "criterion of optimality" chosen.

We now give the reader a taste of the "criterion of optimality" that we will use to compute optimal hedges. Broadly speaking (we say "broadly speaking" because there is a slight twist which we will describe at the end of section (3) - which is why we sometimes refer to "nearly optimal" hedges), our criterion for choosing optimal hedges is by minimising the variance of the terminal value of a self-financing trading strategy designed to hedge the derivative under consideration under a risk-neutral measure $\mathbb{Q}$, henceforth "minimising variance under $\mathbb{Q}$", (for general background, see chapter ten of Cont and Tankov (2004)). This criterion has some advantages. One advantage is that it results in a linear pricing rule which traders might prefer for variance swaps which are very actively traded and are usually considered to be liquid "flow" derivatives (as opposed to highly exotic derivatives). A disadvantage of this criterion is that it weights trading losses and gains equally. A new methodology for pricing and hedging derivatives, termed "pricing and hedging to acceptability", has recently been introduced (see Cherny and Madan (2009), (2010) and Madan (2010) for details) which overcomes this disadvantage. While not the main focus of our paper, we will illustrate numerically how the results obtained from "pricing and hedging to acceptability" are qualitatively broadly in line with results from "minimising variance under $\mathbb{Q}$" which suggests that our results and conclusions have some degree of robustness to the "criterion of optimality" chosen. Both of the "criteria of optimality" that we use model the dynamics under the risk-neutral measure $\mathbb{Q}$. We thank an anonymous referee for pointing out that, since empirical research (see, for example, Broadie et al. (2007)) suggests that, for equity index options, jumps are larger in magnitude and/or more frequent under the risk-neutral measure $\mathbb{Q}$ than under the real-world measure $\mathbb{P}$, the optimal (or nearly optimal) hedges that we derive may be a long way from minimising, for example, the root-mean-square hedging error under $\mathbb{P}$.

In our analysis, we will place a premium on hedging strategies which are relatively simple to implement and have, at least, some degree of robustness to issues such as model mis-specification and to market imperfections or practicalities such as the possible presence, in practice, of transactions costs.

2. Model set-up

Let the initial time (today) be denoted by $t_0 \equiv 0$ and calendar time by t, $t \geq t_0$. We make the following assumption throughout this paper:

Assumption 2.1. Assume that there exists a market, in which a stock trades, which is free of arbitrage. This guarantees the existence of a risk-neutral equivalent martingale measure which will, in general, not be unique since the stock price will typically be assumed to be a process with jumps. If non-unique, assume a risk-neutral equivalent martingale measure, denoted by $\mathbb{Q}$, has been chosen (possibly, via a calibration to market prices). Assume that markets are frictionless. In particular, assume that there are no transactions costs and that continuous trading is possible. Assume that interest-rates and the dividend yield on the stock are deterministic. Assume that we can trade log-forward-contracts.

The ability to trade a continuum of co-terminal vanilla options is a sufficient condition to allow us to trade log-forward-contracts. Since vanilla options will, in practice, not be available for all strikes, this will involve a replication error but we will not discuss this error here since methods for approximate, but, in practice, accurate, replication of log-forward-contracts by co-terminal vanilla options with only a finite set of strikes have already been described in detail in Demeterfi et al. (1999) and in Broadie and Jain (2008).

Note that the assumption above is exactly the same as in the standard 2 + 2 log-contract replication approach of Neuberger (1990). However, unlike in that approach, the stock price is not assumed to have continuous sample paths.

Assumption 2.2. Assume that there exist K independent Lévy processes, denoted by $X_t^{(k)}$, for $k = 1, 2, \ldots, K$, satisfying $X_{t_0}^{(k)} = 0$, each of which is assumed to be mean-corrected so that $\exp(X_t^{(k)})$ is a (non-constant) martingale, under $\mathbb{Q}$, with respect to the natural filtration generated by $X_t^{(k)}$ i.e. that $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t^{(k)})] = \exp(X_{t_0}^{(k)}) = 1$ for all $t \geq t_0$. Assume that for all Υ satisfying $0 \leq \Upsilon \leq 2$, and for all $k = 1, 2, \ldots, K$, $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(\Upsilon X_t^{(k)})]$ and $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(\Upsilon X_t^{(k)})(X_t^{(k)})^n]$ are finite, for some $t > t_0$, for all integers n satisfying $0 \leq n \leq 6$. Let $\overline{\psi}_X^{(k)}(z)$ denote (minus) the (mean-corrected) characteristic exponent of $X_t^{(k)}$, then:

$$\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(izX_t^{(k)})] \equiv \exp(-(t-t_0)\overline{\psi}_X^{(k)}(z)). \tag{1}$$

The assumption of independence means that $X_t^{(k)}$ is independent of $X_t^{(\ell)}$, if $k \neq \ell$.

For each k, let the volatility of the Brownian component $W_t^{(k)}$ (with $W_{t_0}^{(k)} \equiv 0$) of the Lévy process $X_t^{(k)}$ be denoted by $\sigma^{(k)}$, the Lévy measure by $\nu^{(k)}$ and the Poisson random measure (as defined in chapter 2 - see equation 2.90 - of Cont and Tankov (2004)) by $\mu^{(k)}$. By the Lévy-Khinchin formula, for each k, $\overline{\psi}_X^{(k)}(z)$ can be written:

$$-\overline{\psi}_X^{(k)}(z) = -\frac{1}{2}\sigma^{(k)\,2}(z^2 + iz) + \int_{-\infty}^{\infty}(\exp(izx) - 1)\nu^{(k)}(dx) - iz\int_{-\infty}^{\infty}(\exp(x) - 1)\nu^{(k)}(dx). \tag{2}$$

Note that equation (2) applies whether the jump component of $X_t^{(k)}$ has finite activity or infinite activity and whether it has finite or infinite variation (in the latter case, one normally includes a truncation term i.e. the second term in the Lévy-Khinchin formula (for the non-mean-corrected case) is usually written $\int_{-\infty}^{\infty}(\exp(izx) - 1 - izx\mathbf{1}_{|x|<1})\nu^{(k)}(dx)$ but the term $izx\mathbf{1}_{|x|<1}$ always cancels under mean-correction).

For future reference, for each k, the deterministic quantity $m_X^{(k)}(iz)$ is defined via:

$$m_X^{(k)}(iz) \equiv i\overline{\psi}_X^{(k)\prime}(z), \text{ where } ' \text{ denotes differentiation} \tag{3}$$

i.e. $\overline{\psi}_X^{(k)\prime}(z) \equiv \partial\overline{\psi}_X^{(k)}(z)/\partial z$, $\overline{\psi}_X^{(k)\prime\prime}(z) \equiv \partial^2\overline{\psi}_X^{(k)}(z)/\partial z^2$, and further, for $n \geq 3$, $\overline{\psi}_X^{(k),(n)}(z) \equiv \partial^n\overline{\psi}_X^{(k)}(z)/\partial z^n$.

For future reference, we also define, for $t \geq t_0$, and for each k, the following four stochastic processes:

$$\begin{aligned} J_{1,t}^{(k)} &\equiv \int_{t_0}^{t} \Big(\sigma^{(k)\,2} + \int_{-\infty}^{\infty} x^2 \mu^{(k)}(dx)\Big) du, \quad J_{2,t}^{(k)} \equiv \int_{t_0}^{t} \Big(\sigma^{(k)} dW_u^{(k)} + \int_{-\infty}^{\infty} x \mu^{(k)}(dx) du\Big), \\ J_{3,t}^{(k)} &\equiv \int_{t_0}^{t} \int_{-\infty}^{\infty} x^3 \mu^{(k)}(dx) du, \quad J_{4,t}^{(k)} \equiv \int_{t_0}^{t} \Big(\sigma^{(k)} dW_u^{(k)} + \int_{-\infty}^{\infty} (\exp(x) - 1) \mu^{(k)}(dx) du\Big). \end{aligned} \tag{4}$$

In order to construct the stock price process, the Lévy processes $X_t^{(k)}$ will be time-changed so now the time-change processes, denoted by $Y_t^{(k)}$, need to be defined:

Assumption 2.3. Assume that there exist K (possibly, deterministic) non-decreasing, **continuous** time-change processes denoted by $Y_t^{(k)}$, for each $k = 1, 2, \ldots, K$, each of which is a family of stopping times and each of which is of the form $Y_t^{(k)} = \int_{t_0}^t y_s^{(k)} ds$ where the activity rate $y_t^{(k)}$, for each $k = 1, 2, \ldots, K$, must be non-negative. Assume, for each $k = 1, 2, \ldots, K$, that $\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)}] < \infty$ and $Var_{t_0}^{\mathbb{Q}}[Y_T^{(k)}] < \infty$ (for $T < \infty$), that $Y_t^{(k)} \to \infty$ as $t \to \infty$ and that $Y_{t_0}^{(k)} = t_0 \equiv 0$. In general (but see assumption (2.7)), $Y_t^{(k)}$ may be correlated with $X_t^{(\ell)}$, for any $\ell = 1, 2, \ldots, K$.

Remark 2.4. Assumption (2.3) allows the activity rate $y_t^{(k)}$ (defined via $Y_t^{(k)} = \int_{t_0}^t y_s^{(k)} ds$) to follow, for example, a Heston (1993) square-root process, a non-Gaussian OU process (Barndorff-Nielsen and Shephard (2001)) or it could follow the Heston (1993) plus jumps process of Duffie et al. (2000). In the latter two cases, $y_t^{(k)}$ is discontinuous but $Y_t^{(k)}$ is continuous. We stress that it is very important for our analysis that, for each k, $Y_t^{(k)}$ is **continuous** (which, to emphasise even further, rules out $Y_t^{(k)}$ being, for example, a gamma process).

One further set of assumptions is made of a minor technical nature.

Assumption 2.5. Assume, for each $k = 1, 2, \ldots, K$, that for each $\ell = 1, 2, 3, 4$, $\mathbb{E}_{t_0}^{\mathbb{Q}}[J_{\ell,T}^{(k)}] < \infty$ and $Var_{t_0}^{\mathbb{Q}}[J_{\ell,T}^{(k)}] < \infty$ (for $T < \infty$) and, furthermore, that it is possible to decompose $J_{\ell,t}^{(k)}$ into the sum of a martingale process $\tilde{J}_{\ell,t}^{(k)}$ and a drift process $\overline{J}_{\ell,t}^{(k)}$. With analogous notation, a similar assumption is made for $Y_t^{(k)}$, for each $k = 1, 2, \ldots, K$ (such decompositions are always possible (under weak regularity conditions) by the Doob-Meyer decomposition theorem):

$$J_{\ell,t}^{(k)} \equiv \tilde{J}_{\ell,t}^{(k)} + \overline{J}_{\ell,t}^{(k)}, \text{ for each } \ell = 1, 2, 3, 4, \quad Y_t^{(k)} \equiv \tilde{Y}_t^{(k)} + \overline{Y}_t^{(k)}, \text{ for each } k = 1, 2, \ldots, K, \tag{5}$$

where $\tilde{J}_{\ell,t}^{(k)}$ and $\tilde{Y}_t^{(k)}$ are martingales, under $\mathbb{Q}$, with respect to their natural filtrations. In particular, note for future reference that standard results (Cont and Tankov (2004)) imply:

$$\overline{J}_{1,t}^{(k)} = (t - t_0) \overline{\psi}_X^{(k)''}(0), \text{ and } \overline{J}_{3,t}^{(k)} = (t - t_0)(-i \overline{\psi}_X^{(k),(3)}(0)), \text{ for each } k = 1, 2, \ldots, K. \tag{6}$$

Now the price process for the underlying stock is constructed as follows. Let the risk-free interest-rate (respectively, dividend yield on the stock), at time t, be denoted by $r(t)$ (respectively, $q(t)$) (both assumed deterministic and finite). We have K Lévy processes $X_t^{(k)}$ each satisfying assumption (2.2) with $X_{t_0}^{(k)} = 0$. We have K time-change processes $Y_t^{(k)}$ each satisfying assumption (2.3) with $Y_{t_0}^{(k)} = t_0$. For each $k = 1, 2, \ldots, K$, we time-change the Lévy process $X_t^{(k)}$ by $Y_t^{(k)}$ to get a process $X_{Y_t^{(k)}}^{(k)}$ which we henceforth denote by $X_{Y_t}^{(k)}$, with $X_{Y_{t_0}}^{(k)} = 0$.

The stock price $S(t)$, at time t, is assumed to have the following dynamics under $\mathbb{Q}$:

$$S(t) = S(t_0) \exp\left(\int_{t_0}^{t} (r(s) - q(s)) ds\right) \exp\left(\sum_{k=1}^{K} X_{Y_t}^{(k)}\right). \tag{7}$$

Note that $\exp(\sum_{k=1}^{K} X_{Y_t}^{(k)})$ is a martingale, under $\mathbb{Q}$, with respect to the filtration generated by $\mathcal{F}_t \equiv \sigma\{X_{Y_u}^{(1)}, X_{Y_u}^{(2)}, \ldots, X_{Y_u}^{(K)}, u \leq t\}$. Henceforth, whenever we write an expectation in the form $\mathbb{E}_t^{\mathbb{Q}}[\bullet]$, we mean the expectation is conditional on the filtration $\mathcal{F}_t$, at time t.

Remark 2.6. The stock price process assumed in equation (7) allows for many models used in finance including, for example, the VG SAM, CGMY SAM and NIG SAM models of Carr et al. (2003) as well as the jump-diffusion models of Merton (1976) and Kou (2002) (but **not** local volatility type models). In some, if not many, cases of practical interest, it may be that $K = 1$. We have allowed $K > 1$ in order to include the class of stochastic skew models of Carr and Wu (2007) as well as some other models such as the Bates (1996) model and the "double jump model" of Duffie et al. (2000) and of Broadie et al. (2007).

Assumption 2.7. Assumption (2.3) specifies the type of permissible time-change processes in greatest generality. However, a number of our results only apply under more more restrictive assumptions. We will indicate when these more restrictive assumptions are in force by referring to the following:

- 1. It may sometimes be assumed that the time-change processes are common i.e. for all $k = 1, 2, \ldots, K$, $Y_t^{(k)} = Y_t$, say, and $y_t^{(k)} = y_t$, say. This will be referred to as the "common time-change" assumption. Note that when $K = 1$ (which is often the case for models of practical interest), this assumption is not an extra assumption as it must automatically hold.
- 2. It may sometimes be assumed that, for all $k = 1, 2, \ldots, K$ and for all $\ell = 1, 2, \ldots, K$, $X_t^{(\ell)}$ is independent of both $Y_t^{(k)}$ and $y_t^{(k)}$. This will be referred to as the "independent time-change" assumption.
- 3. It may sometimes be assumed that, for all $k = 1, 2, \ldots, K$, the time-change processes $Y_t^{(k)}$ are deterministic (but **not** necessarily of the form $Y_t^{(k)} = t$ and **not** necessarily common). This will be referred to as the "deterministic time-change" assumption.

We will now define the payoffs of the derivatives we will consider in this paper. We will consider derivative contracts, maturing at time T, entered into at time t. We consider continuously monitored variance swaps (henceforth VS), continuously monitored skewness swaps (henceforth SKS), log-forward-contracts (henceforth LFC) whose payoffs, at time T, are (we omit any fixed legs):

$$\lim_{N\to\infty}\sum_{j=1}^{N}(\log(S(u_j)/S(u_{j-1})))^2, \qquad \lim_{N\to\infty}\sum_{j=1}^{N}(\log(S(u_j)/S(u_{j-1})))^3,$$
$$\text{and } \log(F(T,T)/F(t,T)) \equiv \log(S(T)/F(t,T)), \qquad \text{respectively} \tag{8}$$

where $F(t,T) \equiv S(t)\exp(\int_t^T(r(s)-q(s))ds)$ is the forward stock price, at time t, and where $t \equiv u_0 < u_1 < \ldots < u_{j-1} < u_j < \ldots < u_N \equiv T$ defines any partition of $[t,T]$ for which $\sup(u_j - u_{j-1}) \to 0$ as $N \to \infty$ and whose prices, at time t, are respectively denoted by $\mathrm{VS}(t,T)$, $\mathrm{SKS}(t,T)$ and $\mathrm{LFC}(t,T)$. Using results from Carr and Lee (2009) and Crosby and Davis (2010), we have:

$$\begin{aligned}
\mathrm{VS}(t,T) &= P(t,T)\mathbb{E}_t^{\mathbb{Q}}[\sum_{k=1}^{K}\int_t^T y_u^{(k)}\overline{\psi}_X^{(k)''}(0)\,du] = P(t,T)\sum_{k=1}^{K}\overline{\psi}_X^{(k)''}(0)\mathbb{E}_t^{\mathbb{Q}}[Y_T^{(k)} - Y_t^{(k)}],\\
\mathrm{SKS}(t,T) &= -iP(t,T)\sum_{k=1}^{K}\overline{\psi}_X^{(k),(3)}(0)\mathbb{E}_t^{\mathbb{Q}}[Y_T^{(k)} - Y_t^{(k)}],\\
\mathrm{LFC}(t,T) &= P(t,T)\sum_{k=1}^{K}m_X^{(k)}(0)\mathbb{E}_t^{\mathbb{Q}}[Y_T^{(k)} - Y_t^{(k)}], \text{ where } P(t,T) \equiv \exp(-\int_t^T r(s)ds).
\end{aligned} \tag{9}$$

Following Carr and Lee (2009) (and as is also evident from equation (9)), the price, at time t_0, of (the floating leg of) a variance swap is equal to $-Q_X$ times the price, at time t_0, of a log-forward-contract (both maturing at time T), where Q_X is defined by:

$$-Q_X \equiv \frac{\sum_{k=1}^{K}\overline{\psi}_X^{(k)''}(0)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)} - Y_{t_0}^{(k)}]}{\sum_{k=1}^{K}m_X^{(k)}(0)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)} - Y_{t_0}^{(k)}]}. \quad \text{Note that } Q_X > 0, \text{ since } m_X^{(k)} < 0, \text{ for all } k. \tag{10}$$

Remark 2.8. In particular, there is no up-front cost of entering into a position of being long one variance swap (VS) and being long Q_X log-forward-contracts (LFC).

Remark 2.9. Under the "common time-change" assumption, the terms in $\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)} - Y_{t_0}^{(k)}]$ cancel and Q_X depends only upon the parameters of the Lévy processes:
$-Q_X = \sum_{k=1}^{K}\overline{\psi}_X^{(k)''}(0)/\sum_{k=1}^{K}m_X^{(k)}(0)$.

Carr and Lee (2009) show (in a sense they make precise) that Q_X is, respectively, less than two, equal to two or greater than two according as whether the distribution of stock price returns, under $\mathbb{Q}$, is positively skewed, unskewed or negatively skewed (the latter being almost always the case, in practice, for equity markets).

3. Optimal mean-variance quadratic hedging of variance swaps

Our aim is to consider the optimal hedging of variance swaps when the price of the underlying stock has jumps. We will actually consider hedging strategies for variance swaps which fall into three possible types.

The first type of hedging strategy (labelled hedging strategy A) consists of hedging one VS with a static position in LFC and a dynamic position in the underlying stock.

The second type of hedging strategy (labelled hedging strategy B) consists of hedging one VS with a, possibly, dynamic position in LFC as well as a dynamic position in the underlying stock. Whilst a dynamic position in the stock poses no problems, a dynamic position in LFC will involve trading vanilla options, maturing at time T, at all strikes (as already indicated, in practice, only a discrete set of strikes will be available). The bid-offer spread on options will, in practice, be at least one order (perhaps, two orders) of magnitude higher (in price percentage terms) than that on the underlying stock. This will mean, in practice, that a dynamic position in LFC may incur significant transactions costs.

Remark 3.1. Of course, we have assumed from the outset (assumption (2.1)) that there are no transactions costs. However, we wish to have a strategy which is robust to the assumptions made in deriving it.

We will see that, in important special cases, hedging strategy B reduces to a static buy-and-hold position in LFC (as well as dynamic trading in the stock), thus negating practical problems with transactions costs arising from a dynamic position.

The third type of hedging strategy (labelled hedging strategy C) consists of hedging one VS with a, possibly, dynamic position in LFC, a, possibly, dynamic position in SKS as well as a dynamic position in the underlying stock. Skewness swaps (SKS) do not often trade and when they do, they will have a much wider bid-offer spread than log-forward-contracts (LFC). So the same remarks as we made for the second type of hedging strategy (hedging strategy B) also apply here and with even greater validity. However, we will again see that, in important special cases, this strategy also reduces to a static buy-and-hold position in LFC, a static buy-and-hold position in SKS (as well dynamic trading in the stock). We remark that, although we will analyse this type from the point of view of hedging a variance swap with skewness swaps, our analysis would trivially carry over to solve the related problem of optimally hedging a skewness swap with variance swaps. Using skewness swaps to hedge variance swaps was considered in Schoutens (2005) and found to be reasonably effective but we will see that the strategy in Schoutens (2005) is, in general, not optimal, and can be significantly improved upon.

So as to be able to cover all three types of hedging strategy above (A, B and C) without repeating our analysis or rewriting relatively long equations, we will actually initially consider the most general problem of constructing a self-financing trading strategy by taking dynamic positions in VS, in SKS, in LFC and in the underlying stock.

3.1. **Mean-variance quadratic hedging of variance swaps - the general case.** Our aim is to optimally hedge variance swaps. We, initially, consider the problem in most generality from which we will later derive simplified special cases.

We construct a self-financing trading strategy as follows: We commence the strategy at time $t_0 \equiv 0$. At each time $t \in [t_0, T]$, we hold a position in Θ_t^{VS} VS, in Θ_t^{LFC} LFC and in Θ_t^{SKS} SKS. Additionally, we trade dynamically in the underlying stock. Specifically, for all $t \in [t_0, T]$, we hold a short position in Δ_t units of stock. We state the value $\epsilon(T)$, at time T, of our self-financing trading strategy in the following proposition:

Proposition 3.2. *The value $\epsilon(T)$, at time T, of our self-financing trading strategy is:*

$$\epsilon(T) \equiv \epsilon_L(T) + \epsilon_C(T), \tag{11}$$

where (the subscripts "L" and "C" are mnemonics for "Lévy" and "clock" respectively)

$$\begin{aligned} \epsilon_L(T) \equiv & \int_{t_0}^{T} \Theta_u^{\mathrm{VS}} \sum_{k=1}^{K} d\tilde{J}_{1,Y_u}^{(k)} + \int_{t_0}^{T} \Theta_u^{\mathrm{LFC}} \sum_{k=1}^{K} d\tilde{J}_{2,Y_u}^{(k)} + \int_{t_0}^{T} \Theta_u^{\mathrm{SKS}} \sum_{k=1}^{K} d\tilde{J}_{3,Y_u}^{(k)} \\ & - \int_{t_0}^{T} \Delta_u S(u-) \sum_{k=1}^{K} d\tilde{J}_{4,Y_u}^{(k)}, \quad \textit{where, for each } \ell = 1,2,3,4,\ \tilde{J}_{\ell,Y_t}^{(k)} \equiv \tilde{J}_{\ell,Y_t^{(k)}}^{(k)},\ J_{\ell,Y_t}^{(k)} \equiv J_{\ell,Y_t^{(k)}}^{(k)} \end{aligned} \tag{12}$$

$$\textit{and} \quad \epsilon_C(T) \equiv \int_{t_0}^{T} \sum_{k=1}^{K} \Big(\Theta_u^{\mathrm{VS}} \overline{\psi}_X^{(k)\prime\prime}(0) + \Theta_u^{\mathrm{LFC}} m_X^{(k)}(0) + \Theta_u^{\mathrm{SKS}} (-i \overline{\psi}_X^{(k),(3)}(0)) \Big) d\tilde{Y}_u^{(k)}. \tag{13}$$

Proof: Firstly, define a money market account $B(t) = \exp(\int_{t_0}^{t} r(s)ds)$, $t \geq t_0$, and denote the position in it, at time t, by Θ_t^B. The zero net aggregate investment condition, at time u, (for $u \in [t_0, T]$) reads:

$$0 = \Theta_u^B B(u) + \Theta_u^{\mathrm{VS}} \mathrm{VS}(u,T) + \Theta_u^{\mathrm{LFC}} \mathrm{LFC}(u,T) + \Theta_u^{\mathrm{SKS}} \mathrm{SKS}(u,T) - \Delta_u S(u-). \tag{14}$$

The easiest way to proceed is to note the additive nature of the VS, LFC and SKS payoffs and to consider the contribution to $\epsilon(T)$ over an infinitesimal time period u to $u + du$. The profit-and-loss over the time period u to $u + du$ is:

$$\begin{aligned} & \Theta_u^B r(u) B(u) du + \Theta_u^{\mathrm{VS}} \Big(r(u) \mathrm{VS}(u,T) du + \sum_{k=1}^{K} dJ_{1,Y_u}^{(k)} \Big) \\ + \ & \Theta_u^{\mathrm{LFC}} \Big(\sum_{k=1}^{K} \Big(r(u) \mathrm{LFC}(u,T) du + d\tilde{J}_{2,Y_u}^{(k)} + m_X^{(k)}(0) du \Big) \Big) \\ + \ & \Theta_u^{\mathrm{SKS}} \Big(r(u) \mathrm{SKS}(u,T) du + \sum_{k=1}^{K} dJ_{3,Y_u}^{(k)} \Big) - \Delta_u S(u-) \Big((r(u) - q(u)) du + \sum_{k=1}^{K} d\tilde{J}_{4,Y_u}^{(k)} + q(u) du \Big). \end{aligned}$$

In the above, we have used Ito's lemma. The final term involving Δ_u is the profit-and-loss from the position in the stock (with the final term $q(u)du$ resulting from re-invested dividends). Substituting from equation (14), the profit-and-loss over the time period u to $u+du$ is:

$$\begin{aligned}
& \Theta_u^{\mathrm{VS}} \sum_{k=1}^{K} dJ_{1,Y_u}^{(k)} + \Theta_u^{\mathrm{LFC}} \sum_{k=1}^{K} \left(d\tilde{J}_{2,Y_u}^{(k)} + m_X^{(k)}(0) \right) + \Theta_u^{\mathrm{SKS}} \sum_{k=1}^{K} dJ_{3,Y_u}^{(k)} \\
- \; & \Delta_u S(u-) \sum_{k=1}^{K} d\tilde{J}_{4,Y_u}^{(k)} \\
= \; & \Theta_u^{\mathrm{VS}} \sum_{k=1}^{K} d\tilde{J}_{1,Y_u}^{(k)} + \Theta_u^{\mathrm{LFC}} \sum_{k=1}^{K} d\tilde{J}_{2,Y_u}^{(k)} + \Theta_u^{\mathrm{SKS}} \sum_{k=1}^{K} d\tilde{J}_{3,Y_u}^{(k)} - \Delta_u S(u-) \sum_{k=1}^{K} d\tilde{J}_{4,Y_u}^{(k)} \\
+ \; & \sum_{k=1}^{K} \left(\Theta_u^{\mathrm{VS}} \overline{\psi}_X^{(k)''}(0) + \Theta_u^{\mathrm{LFC}} m_X^{(k)}(0) + \Theta_u^{\mathrm{SKS}} (-i\overline{\psi}_X^{(k),(3)}(0)) \right) d\tilde{Y}_u^{(k)}, \qquad (15)
\end{aligned}$$

where we have used equation (6). The result now follows by integrating from t_0 to T. •

It is straightforward to see that, for any values of Θ_t^{VS}, Θ_t^{LFC}, Θ_t^{SKS} and Δ_t, we have: $\mathbb{E}_{t_0}^{\mathbb{Q}}[\epsilon_L(T)] = 0$, $\mathbb{E}_{t_0}^{\mathbb{Q}}[\epsilon_C(T)] = 0$ and $\mathbb{E}_{t_0}^{\mathbb{Q}}[\epsilon(T)] = 0$ which is very intuitive.

Broadly speaking, our aim will be to find the (assumed bounded) predictable processes Θ_t^{VS}, Θ_t^{LFC}, Θ_t^{SKS} and Δ_t such that the variance (under $\mathbb{Q}$) of the residual hedging error is minimised. With the aid of Ito's isometry formula, we can calculate the variance, $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$, under $\mathbb{Q}$, of the value $\epsilon(T)$ of our self-financing trading strategy, which we state in the following proposition:

Proposition 3.3. *Define*

$$\begin{aligned}
\phi_t \; & \equiv \; \Delta_t S(t-), \quad \textit{then:} \\
Var_{t_0}^{\mathbb{Q}}[\epsilon(T)] \; & = \; Var_{t_0}^{\mathbb{Q}}[\epsilon_C(T)] + 2\, Covar_{t_0}^{\mathbb{Q}}[\epsilon_C(T), \epsilon_L(T)] + Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)], \qquad (16)
\end{aligned}$$

where $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$ has the explicit form:

$$\begin{aligned}
Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)] \; = \; & \sum_{k=1}^{K} \mathbb{E}_{t_0}^{\mathbb{Q}}[\int_{t_0}^{T} y_u^{(k)} \Big(\phi_u^2 (-\overline{\psi}_X^{(k)}(-2i)) - 2\phi_u \Big(\Theta_u^{\mathrm{LFC}} (m_X^{(k)}(1) - m_X^{(k)}(0)) \\
+ \; & \Theta_u^{\mathrm{SKS}} i(\overline{\psi}_X^{(k),(3)}(-i) - \overline{\psi}_X^{(k),(3)}(0)) + \Theta_u^{\mathrm{VS}} (\overline{\psi}_X^{(k)''}(-i) - \overline{\psi}_X^{(k)''}(0)) \Big) \\
+ \; & \Big(\Theta_u^{\mathrm{SKS}\,2} \overline{\psi}_X^{(k),(6)}(0) + 2\Theta_u^{\mathrm{SKS}} \Theta_u^{\mathrm{VS}} i\overline{\psi}_X^{(k),(5)}(0) - 2\Theta_u^{\mathrm{SKS}} \Theta_u^{\mathrm{LFC}} \overline{\psi}_X^{(k),(4)}(0) \\
- \; & \Theta_u^{\mathrm{VS}\,2} \overline{\psi}_X^{(k),(4)}(0) - 2\Theta_u^{\mathrm{VS}} \Theta_u^{\mathrm{LFC}} i\overline{\psi}_X^{(k),(3)}(0) + \Theta_u^{\mathrm{LFC}\,2} \overline{\psi}_X^{(k)''}(0) \Big) \Big) du]. \qquad (17)
\end{aligned}$$

To state more explicit forms for $Var_{t_0}^{\mathbb{Q}}[\epsilon_C(T)]$ and for $Covar_{t_0}^{\mathbb{Q}}[\epsilon_C(T), \epsilon_L(T)]$, it is useful to define:

$$\begin{aligned}
& \Theta_u^{(1)} \equiv \Theta_u^{\mathrm{VS}}, \quad \Theta_u^{(2)} \equiv \Theta_u^{\mathrm{LFC}}, \quad \Theta_u^{(3)} \equiv \Theta_u^{\mathrm{SKS}}, \; \textit{and for each } k = 1, 2, \ldots, K, \\
& \alpha_{1,k} \equiv \overline{\psi}_X^{(k)''}(0), \quad \alpha_{2,k} \equiv m_X^{(k)}(0), \quad \alpha_{3,k} \equiv (-i\overline{\psi}_X^{(k),(3)}(0)). \qquad (18)
\end{aligned}$$

Then using square brackets to denote quadratic covariation:

$$Var^{\mathbb{Q}}_{t_0}[\epsilon_C(T)] = \mathbb{E}^{\mathbb{Q}}_{t_0}[\int_{t_0}^{T}\sum_{i=1}^{3}\sum_{j=1}^{3}\Theta^{(i)}_u\Theta^{(j)}_u\sum_{k=1}^{K}\sum_{\ell=1}^{K}\alpha_{i,k}\alpha_{j,\ell}\big[\tilde{Y}^{(k)}_{\bullet},\tilde{Y}^{(\ell)}_{\bullet}\big]_u du], \text{ and} \tag{19}$$

$$\begin{aligned} Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T),\epsilon_L(T)] &= \mathbb{E}^{\mathbb{Q}}_{t_0}[\int_{t_0}^{T}\sum_{i=1}^{3}\sum_{j=1}^{3}\Theta^{(i)}_u\Theta^{(j)}_u\sum_{k=1}^{K}\sum_{\ell=1}^{K}\alpha_{i,k}\big[\tilde{J}^{(k)}_{i,Y_{\bullet}},\tilde{Y}^{(\ell)}_{\bullet}\big]_u du \\ &\quad - \int_{t_0}^{T}\sum_{i=1}^{3}\Theta^{(i)}_u\phi_u\sum_{k=1}^{K}\sum_{\ell=1}^{K}\alpha_{i,k}\big[\tilde{J}^{(k)}_{4,Y_{\bullet}},\tilde{Y}^{(\ell)}_{\bullet}\big]_u du]. \end{aligned} \tag{20}$$

Proof: Use Ito's isometry formula and then simplify using equation (2). •

Remark 3.4. Note that equation (16) implies that $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ is a non-negative quadratic function of Θ^{VS}_t, Θ^{LFC}_t, Θ^{SKS}_t and Δ_t (and ϕ_t). Note further that equations (11) and (16) apply for any choices of the (assumed bounded) predictable processes Θ^{VS}_t, Θ^{LFC}_t, Θ^{SKS}_t and Δ_t.

For reasons that will become clearer later, in cases of practical interest, $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ can often be evaluated simply by computing the first six derivatives of $\overline{\psi}^{(k)}_X(z)$ as well as $\overline{\psi}^{(k)}_X(-2i)$, for each k, which is typically (eg. CGMY (Carr et al. (2003)), Kou (2002) double-exponential jump-diffusion model) trivial and by computing $\mathbb{E}^{\mathbb{Q}}_{t_0}[\int_{t_0}^{T} y^{(k)}_u du] = \mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T - Y^{(k)}_{t_0}]$ which is often (for example, in those cases cited in remark (2.4)) known in closed form.

Note that $Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T),\epsilon_L(T)]$ will be identically equal to zero under the "independent time-change" assumption. Furthermore, note that both $Var^{\mathbb{Q}}_{t_0}[\epsilon_C(T)]$ and $Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T),\epsilon_L(T)]$ will be identically equal to zero if either:

$$\Theta^{\mathrm{VS}}_u\overline{\psi}^{(k)\prime\prime}_X(0) + \Theta^{\mathrm{LFC}}_u m^{(k)}_X(0) + \Theta^{\mathrm{SKS}}_u(-i\overline{\psi}^{(k),(3)}_X(0)) = 0, \text{ for all } k \text{ and for all } u\in[t_0,T], \tag{21}$$

or if the "common time-change" assumptions holds and

$$\sum_{k=1}^{K}\big(\Theta^{\mathrm{VS}}_u\overline{\psi}^{(k)\prime\prime}_X(0) + \Theta^{\mathrm{LFC}}_u m^{(k)}_X(0) + \Theta^{\mathrm{SKS}}_u(-i\overline{\psi}^{(k),(3)}_X(0))\big) = 0, \text{ for all } u\in[t_0,T], \tag{22}$$

or if the "deterministic time-change" assumption holds.

The above equations present our problem in greatest generality but issues, already remarked upon, of liquidity, availability of hedging instruments and transactions costs (see remark (3.1)) may mean that we should look for simplifications. We will consider the problem from the point of view of hedging a static position in variance swaps - more precisely, we hedge a static position of being long one VS, so we set $\Theta^{\mathrm{VS}}_t = 1$, for all $t\in[t_0,T]$. We will examine hedging strategies A, B and C in detail. We will wish to find the values, at time t, of the portfolio weights ($\Delta_t \equiv \phi_t/S(t-)$ and, when appropriate, Θ^{LFC}_t and Θ^{SKS}_t) which, broadly speaking, minimise the variance of our self-financing trading strategy and we can do this by differentiating the variance with respect to the relevant portfolio weight(s) and setting the resulting equation(s) to zero. We will use the notation that when we have computed the optimal values, we will denote them by $\hat{\Delta}_t \equiv \hat{\phi}_t/S(t-)$ (and, when appropriate, $\hat{\Theta}^{\mathrm{LFC}}_t$ and $\hat{\Theta}^{\mathrm{SKS}}_t$) i.e. by using the hat symbol.

It is certainly possible to differentiate equation (16) as written above in full generality. However, partly because the forms of $Var^{\mathbb{Q}}_{t_0}[\epsilon_C(T)]$ and $Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T),\epsilon_L(T)]$ are not very explicit and partly for reasons that will become apparent later, what we will actually do is consider two possible variants, which we label variants (a) and (b).

• In variant (a), which we describe in detail, for each of the hedging strategies A, B and C, in section (4), we will actually seek to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$. What we will see is that, in some cases of interest, it turns out that $Var^{\mathbb{Q}}_{t_0}[\epsilon_C(T)]$ and $Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T),\epsilon_L(T)]$ happen to be equal to zero and hence minimising $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ results in a value of $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ equal to $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$. Additionally, in an important special case, hedging strategies B and C will be seen to reduce to static buy-and-hold positions in LFC and SKS (hedging strategy A is always so by construction).

• In variant (b), which we describe in section (5), we seek to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$. This variant is theoretically much more appealing, of course - after all, it is $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ (and not $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$) which is a measure of the total residual hedging error. However, what we will see is that there is a major practical disadvantage to this variant - namely that (except under the "deterministic time-change" assumption - and even then only when the "common time-change" assumption additionally holds), hedging strategies B and C never reduce to static buy-and-hold positions in LFC or SKS which makes these strategies vulnerable, in practice, to the impact of transactions costs (see remark (3.1)). Clearly, under the "deterministic time-change" assumption, variants (a) and (b) amount to the same thing.

4. Variant (a): Minimising $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$

In this section, we will consider variant (a) i.e. the problem of minimising $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ (defined via equations (12) and (17)), for each of the three hedging strategies A, B and C (defined in section (3)). We set $\Theta^{\text{VS}}_t = 1$ throughout. Essentially, we differentiate $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ (given by equation (17)) with respect to ϕ_t, with respect to Θ^{LFC}_t and with respect to Θ^{SKS}_t and set the resulting equations to zero. We obtain:

$$
\begin{aligned}
-\sum_{k=1}^{K} y_t^{(k)}(\overline{\psi}_X^{(k)\prime\prime}(-i)-\overline{\psi}_X^{(k)\prime\prime}(0)) \quad &= \quad \hat{\Theta}^{\text{LFC}}_t\Big(\sum_{k=1}^{K} y_t^{(k)}(m_X^{(k)}(1)-m_X^{(k)}(0))\Big)+\hat{\phi}_t\Big(\sum_{k=1}^{K} y_t^{(k)}\overline{\psi}_X^{(k)}(-2i)\Big) \\
&- \quad \hat{\Theta}^{\text{SKS}}_t\Big(\sum_{k=1}^{K} y_t^{(k)}(i\overline{\psi}_X^{(k),(3)}(-i)-i\overline{\psi}_X^{(k),(3)}(0))\Big), \\
\sum_{k=1}^{K} y_t^{(k)} i\overline{\psi}_X^{(k),(3)}(0) \quad &= \quad \hat{\Theta}^{\text{LFC}}_t\Big(\sum_{k=1}^{K} y_t^{(k)}\overline{\psi}_X^{(k)\prime\prime}(0)\Big)-\hat{\phi}_t\Big(\sum_{k=1}^{K} y_t^{(k)}(m_X^{(k)}(1)-m_X^{(k)}(0))\Big) \\
&- \quad \hat{\Theta}^{\text{SKS}}_t\Big(\sum_{k=1}^{K} y_t^{(k)}\overline{\psi}_X^{(k),(4)}(0)\Big),
\end{aligned}
$$

$$
\begin{aligned}
-\sum_{k=1}^{K} y_t^{(k)} i\overline{\psi}_X^{(k),(5)}(0) &= -\hat{\Theta}_t^{\mathrm{LFC}}\Big(\sum_{k=1}^{K} y_t^{(k)}\overline{\psi}_X^{(k),(4)}(0)\Big) + \hat{\phi}_t\Big(\sum_{k=1}^{K} y_t^{(k)}(i\overline{\psi}_X^{(k),(3)}(-i) - i\overline{\psi}_X^{(k),(3)}(0))\Big) \\
&+ \hat{\Theta}_t^{\mathrm{SKS}}\Big(\sum_{k=1}^{K} y_t^{(k)}\overline{\psi}_X^{(k),(6)}(0)\Big). \tag{23}
\end{aligned}
$$

This gives us a $3X3$ system of simultaneous **linear** equations (in general - for hedging strategies A and B they reduce to $1X1$ and $2X2$ respectively). Provided the determinant of the linear system is non-zero, we can solve for the optimal values $\hat{\phi}_t$, $\hat{\Theta}_t^{\mathrm{LFC}}$ and $\hat{\Theta}_t^{\mathrm{SKS}}$. We do not give the explicit solutions since they are quite long and it would add little in intuition. However, even without writing down the explicit solutions, it is clear from casual inspection of equation (23) that, whenever the "common time-change" assumption is in force, the $y_t^{(k)}$ terms cancel throughout.

Remark 4.1. The fact that the $y_t^{(k)}$ terms cancel throughout implies that $\hat{\Theta}_t^{\mathrm{LFC}}$, $\hat{\Theta}_t^{\mathrm{SKS}}$ and $\hat{\phi}_t$ are constant and, so in particular, hedging strategies A, B and C all reduce to a static buy-and-hold position in LFC and in SKS. As already alluded to (see remark (3.1)), static buy-and-hold (as opposed to dynamic) positions imply a degree of robustness to the possible presence, in practice, of transactions costs. Furthermore, the optimal values $\hat{\Theta}_t^{\mathrm{LFC}}$, $\hat{\Theta}_t^{\mathrm{SKS}}$ and $\hat{\phi}_t$ do not depend upon the time-change process in any way which gives a considerable degree of robustness to model mis-specification (of the time-change). We stress that the conclusions contained within the confines of this remark (while certainly requiring the "common time-change" assumption) are valid **without** assuming that the "independent time-change" or the "deterministic time-change" assumptions are in force. This parallels results in Carr and Lee (2009) (see also remark (2.9)).

4.1. **Hedging strategy A: Hedging variance swaps with a static position in log-forward-contracts.** We now consider hedging strategy A in more detail. Since we do not use skewness swaps for this hedging strategy, we set $\Theta_t^{\mathrm{SKS}} = 0$, for all $t \in [t_0, T]$. Additionally, we wish to have only a static buy-and-hold position in LFC. Motivated by equation (10) and remark (2.8), we choose $\Theta_t^{\mathrm{LFC}} = Q_X$, for all t. To find the portfolio which minimises $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$, we can simply solve the first sub-equation of equation (23) for $\hat{\phi}_t$. We find that the optimal value $\hat{\phi}_t \equiv \hat{\Delta}_t S(t-)$ which minimises $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$ is:

$$
\hat{\phi}_t = \frac{\sum_{k=1}^{K} y_t^{(k)}(\overline{\psi}_X^{(k)''}(-i) - \overline{\psi}_X^{(k)''}(0)) + Q_X \sum_{k=1}^{K} y_t^{(k)}(m_X^{(k)}(1) - m_X^{(k)}(0))}{-\sum_{k=1}^{K} y_t^{(k)}\overline{\psi}_X^{(k)}(-2i)}. \tag{24}
$$

Note that under the "common time-change" assumption (which must automatically hold when $K = 1$), we can see (using equation (10) which implies that $Q_X = -\sum_{k=1}^{K}\overline{\psi}_X^{(k)''}(0)/\sum_{k=1}^{K} m_X^{(k)}(0)$ in this special case) that equation (22) is satisfied. This means that:

$Var_{t_0}^{\mathbb{Q}}[\epsilon_C(T)] = Covar_{t_0}^{\mathbb{Q}}[\epsilon_C(T), \epsilon_L(T)] = 0$ and the optimal value $\hat{\phi}_t$ in equation (24) not only minimises $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$ - it results in a value of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ equal to $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$.

Remark 4.2. We now consider equation (24) in two special cases, labelled A1 and A2.

First case A1:

The first special case is when all K Lévy processes are Brownian motions. We find that $Q_X = 2$ and $\hat{\phi}_t = Q_X = 2$ which agrees with standard results (Neuberger (1990)). Clearly, $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)] = 0$ in this special case and furthermore (from equation (21)) $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)] = 0$ and so we have a perfect hedge.

Second case A2:

The second special case is when $K = 1$ and the Lévy process is a compound Poisson process (with **no** diffusion component), with intensity rate λ_1 under $\mathbb{Q}$ and with a fixed jump amplitude a_1. We have $\overline{\psi}^{(1)}_X(z) = -\lambda_1(\exp(iza_1) - 1 - iz\exp(a_1) + iz)$, $m^{(1)}_X(0) = -\lambda_1(\exp(a_1) - 1 - a_1)$ and $\overline{\psi}^{(1)''}_X(0) = \lambda_1 a_1^2$. Hence $Q_X \equiv -\overline{\psi}^{(1)''}_X(0)/m^{(1)}_X(0) = a_1^2/(\exp(a_1) - 1 - a_1)$. Substituting, we get: $\hat{\phi}_t = a_1^2/(\exp(a_1) - 1 - a_1)$. We see that $\hat{\phi}_t = Q_X$. It is straightforward to verify that $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)] = 0$ and that, furthermore, (from equation (22)) $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)] = 0$ in this special case - in other words, we have a perfect hedge.

It is well-known that when the log of the stock price follows either Brownian motion or a compound Poisson process with a fixed jump amplitude, then the market is complete and it is possible to perfectly hedge all contingent claims. Here we can also allow for a stochastic time-change (or indeed multiple stochastic time-changes for special case A1) and hence our market is, in general, not complete. It is not possible to perfectly hedge all contingent claims but it is possible to hedge variance swaps perfectly by taking a static buy-and-hold position in LFC and a dynamic position in the underlying stock in the two special cases A1 and A2 just outlined.

It is worthy of note that in both these special cases A1 and A2, the optimal value $\hat{\phi}_t$ which minimises $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ is equal to Q_X and hence also equal to Θ^{LFC}_t.

It is also interesting to consider what happens in special case A2 when the fixed jump amplitude a_1 is very small. Then, expanding the exp function in a power series:

$$\hat{\phi}_t = Q_X = \frac{a_1^2}{(\exp(a_1) - 1 - a_1)} \approx \frac{2}{(1 + (a_1/3))}.$$

We see that when a_1 is small but positive, $\hat{\phi}_t = Q_X$ is just below two and when a_1 is small but negative, $\hat{\phi}_t = Q_X$ is just above two. In either case, as $a_1 \to 0$, $Q_X \to 2$, $\phi_t \to 2$, which is the same as the case of Brownian motion. We can see that the standard $2 + 2$ log-contract replication approach naturally appears as the "small jump limit" of our more general analysis.

4.2. **Hedging strategy B: Hedging variance swaps with a (possibly) dynamic position in log-forward-contracts.** We now consider hedging strategy B. To find the portfolio which minimises $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$, we can simply solve the first two sub-equations of equation (23) (with $\Theta^{\text{SKS}}_t = 0$ for all t) for $\hat{\phi}_t$ and $\hat{\Theta}^{\text{LFC}}_t$.

For brevity, we do not give the explicit solutions in the general case here since they are quite long but they are trivial to compute numerically.

Remark 4.3. We will however briefly consider the solution in two special cases, labelled B1 and B2. In both these special cases, we make the "common time-change" and the "deterministic time-change" assumptions.

First case B1:

The first special case is when one of the Lévy processes is a compound Poisson process, with intensity rate λ_1 under $\mathbb{Q}$ and with a fixed jump amplitude a_1, and the remaining $K-1$ Lévy processes are Brownian motions (which could be combined into a single Brownian motion in the obvious way). After substituting the relevant characteristic functions, we find that the values $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$ which minimise $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$ are:

$$\hat{\phi}_t = \frac{a_1^2}{(\exp(a_1) - 1 - a_1)}, \quad \text{and} \quad \hat{\Theta}_t^{\mathrm{LFC}} = \frac{a_1^2}{(\exp(a_1) - 1 - a_1)}. \tag{25}$$

Note that these values are of the same form as those obtained in special case A2 and that $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$ are equal and they do not depend on λ_1 or on the volatilities associated with the Brownian motions.

Second case B2:

The second special case is when $K = 2$ and the Lévy processes are both compound Poisson processes (with **no** diffusion component) (whose intensity rates, under $\mathbb{Q}$, are denoted by λ_1 and λ_2) with fixed jump amplitudes (which we denote by a_1 and a_2 respectively, with $a_1 \neq a_2$ (to avoid a degeneracy)). After some straightforward algebra, we find:

$$\hat{\phi}_t = \frac{a_1^2 a_2 - a_1 a_2^2}{a_2(\exp(a_1) - 1) - a_1(\exp(a_2) - 1)}, \quad \text{and} \ \hat{\Theta}_t^{\mathrm{LFC}} = \frac{a_1^2(\exp(a_2) - 1) - a_2^2(\exp(a_1) - 1)}{a_2(\exp(a_1) - 1) - a_1(\exp(a_2) - 1)}.$$

Note that $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$ do not depend on λ_1 or on λ_2.

It is straightforward (if a little tedious) to verify that, in both our special cases B1 and B2, that, if we compute the variance $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ of our self-financing trading strategy, with the respective values of $\hat{\Theta}_t^{\mathrm{LFC}}$ and $\hat{\phi}_t$, we find that $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)] = 0$ and hence $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)] = 0$ - in other words, we again have a perfect hedge.

For the second case, we also consider the following limiting case. We set $a_1 = a$, $a_2 = -a$, where $a > 0$ is very small. Then, expanding the exp function in a power series, we find after some algebra that:

$$\hat{\Theta}_t^{\mathrm{LFC}} \approx \frac{2(1 + \frac{a^2}{6})}{1 + \frac{a^2}{12}} \approx 2(1 + \frac{a^2}{12}), \quad \hat{\phi}_t \approx \frac{2}{1 + \frac{a^2}{12}} \approx 2(1 - \frac{a^2}{12}).$$

As $a \to 0$, $\hat{\Theta}_t^{\mathrm{LFC}} \to 2$ and $\hat{\phi}_t \to 2$, which is the same as the case of Brownian motion.

4.3. **Hedging strategy C: Hedging variance swaps with a (possibly) dynamic position in log-forward-contracts and skewness swaps.** We now briefly consider hedging strategy C.

Remark 4.4. We will briefly consider the solution in two special cases, labelled C1 and C2. In both these special cases, we make the "common time-change" and the "deterministic time-change" assumptions.

First case C1:

The first special case is when two of the Lévy processes are compound Poisson processes, with fixed jump amplitudes a_1 and a_2 repectively (with $a_1 \neq a_2$ (to avoid a degeneracy)), and the remaining $K-2$ Lévy processes are Brownian motions (which could be combined into a single Brownian motion in the obvious way). For brevity, we do not write down the explicit solutions for $\hat{\Theta}_t^{\mathrm{LFC}}$ and $\hat{\phi}_t$ but the optimal value $\hat{\Theta}_t^{\mathrm{SKS}}$ can be shown with some algebra, to be given by:

$$\hat{\Theta}_t^{\mathrm{SKS}} = \frac{a_1^2(\exp(a_2)-1-a_2)-a_2^2(\exp(a_1)-1-a_1)}{a_2^3(\exp(a_1)-1-a_1)-a_1^3(\exp(a_2)-1-a_2)}.$$

We also consider the following limiting case. We set $a_1 = a$, $a_2 = -a$, where $a > 0$ is very small. Then, expanding the exp function in a power series, we get:

$$\hat{\Theta}_t^{\mathrm{SKS}} \approx \frac{1}{3}\left(\frac{1+\frac{a^2}{20}}{1+\frac{a^2}{12}}\right) \approx \frac{1}{3}\left(1-\frac{a^2}{30}\right).$$

Hence as $a \to 0$, $\hat{\Theta}_t^{\mathrm{SKS}} \to 1/3$ which can be compared to a result in Schoutens (2005) obtained by a completely different methodology. In the same limit, we can also show that $\hat{\phi}_t \to 2$, $\hat{\Theta}_t^{\mathrm{LFC}} \to 2$.

Second case C2:

The second special case is when $K = 3$ and the Lévy processes are all compound Poisson processes (with **no** diffusion component), with fixed jump amplitudes (which we denote by a_1, a_2 and a_3 respectively, with $a_1 \neq a_2 \neq a_3$ (to avoid a degeneracy)). The explicit solutions are omitted for brevity but, it can be shown that, in the limit that $|a_1| \to 0$, $|a_2| \to 0$ and $|a_3| \to 0$, then $\hat{\Theta}_t^{\mathrm{SKS}} \to 1/3$, and $\hat{\phi}_t \to 2$, $\hat{\Theta}_t^{\mathrm{LFC}} \to 2$.

It is straightforward (if a little tedious) to verify that, in both our special cases C1 and C2, that, if we compute the variance $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ of our self-financing trading strategy, with the respective values of $\hat{\Theta}_t^{\mathrm{LFC}}$, $\hat{\Theta}_t^{\mathrm{SKS}}$ and $\hat{\phi}_t$, we find that $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)] = 0$ and hence $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)] = 0$ - in other words, in both special cases C1 and C2, we again have a perfect hedge.

We also see that the approach of Schoutens (2005), which essentially sets $\phi_t = 2$, $\Theta_t^{\mathrm{LFC}} = 2$ and $\Theta_t^{\mathrm{SKS}} = 1/3$, naturally appears as the "small jump limit" of our more general analysis.

5. Variant (b): Minimising $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$

In section (4), we considered variant (a) i.e. the problem of minimising $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$. Of course, it would be theoretically more appealing to seek to minimise $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ rather than $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$. In this section, we will now consider the impact of seeking to minimise $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ on hedging strategy A and hedging strategy B (we could do likewise for hedging strategy C but we will omit this case for brevity). We set $\Theta_t^{\mathrm{VS}} = 1$, $\Theta_t^{\mathrm{SKS}} = 0$, for all $t \in [t_0, T]$, in equation (16).

5.1. **Hedging strategy A for the case of minimising** $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$**.** We again (motivated by remark (2.8)) choose $\Theta_t^{\mathrm{LFC}} = Q_X$, for all $t \in [t_0, T]$, in equation (16). We then differentiate it with respect to ϕ_t and set the resulting equation to zero. We find that the optimal value $\hat{\phi}_t$ which minimises $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ is:

$$\begin{aligned} \hat{\phi}_t &= \frac{\sum_{k=1}^K y_t^{(k)} (\overline{\psi}_X^{(k)''}(-i) - \overline{\psi}_X^{(k)''}(0)) + Q_X \sum_{k=1}^K y_t^{(k)} (m_X^{(k)}(1) - m_X^{(k)}(0))}{-\sum_{k=1}^K y_t^{(k)} \overline{\psi}_X^{(k)}(-2i)} \\ &+ \frac{\sum_{k=1}^K (\overline{\psi}_X^{(k)''}(0) + Q_X\, m_X^{(k)}(0)) \sum_{j=1}^K \left[\tilde{J}_{4,Y_\bullet}^{(j)}, \tilde{Y}_\bullet^{(k)}\right]_t}{-\sum_{k=1}^K y_t^{(k)} \overline{\psi}_X^{(k)}(-2i)}. \end{aligned} \tag{26}$$

The first term in equation (26) is the same as in equation (24). However, now we have a second term. Analogously to the argument we used immediately after equation (24), note that under the "common time-change" assumption (which must automatically hold when $K = 1$), we can see (by substituting the value of Q_X - see remark (2.9)) that the numerator of the second term vanishes and so our expression for $\hat{\phi}_t$ in equation (26) is the same as that in equation (24). Note further, that under the "independent time-change" assumption (and, therefore, also, under the "deterministic time-change" assumption) the numerator of the second term in equation (26) vanishes even if the "common time-change" assumption is not in force. Therefore, we see that under the "common time-change" assumption or under the "independent time-change" assumption, the value of $\hat{\phi}_t$ obtained by variant (b) (i.e. by minimising $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$) is the same as the value of $\hat{\phi}_t$ obtained by variant (a) (i.e. by minimising $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$).

5.2. **Hedging strategy B for the case of minimising** $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$**.** Our aim is to find Θ_t^{LFC} and $\Delta_t \equiv \phi_t / S(t-)$ such that we minimise $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$. We differentiate equation (16) with respect to ϕ_t and with respect to Θ_t^{LFC}, set the resulting equations to zero and solve for the optimal values $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$. This only involves solving a $2X2$ linear system but since the resulting equations are quite long, we do not write down the solution in its most explicit form. However, even casual inspection of equation (16) tells us that the resulting solution will be in terms of $y_t^{(k)}$, $\left[\tilde{Y}_\bullet^{(k)}, \tilde{Y}_\bullet^{(j)}\right]_t$ and $\left[\tilde{J}_{\ell,Y_\bullet}^{(j)}, \tilde{Y}_\bullet^{(k)}\right]_t$, for $\ell = 1, 2, 3, 4$ and for all $k, j = 1, 2, \ldots, K$. Two important comments follow: Firstly, to the extent that $\left[\tilde{Y}_\bullet^{(k)}, \tilde{Y}_\bullet^{(j)}\right]_t$, and $\left[\tilde{J}_{\ell,Y_\bullet}^{(j)}, \tilde{Y}_\bullet^{(k)}\right]_t$ do not have explicit forms for completely general stochastic time-change processes, the resulting solution will not be useful. Secondly, and more importantly, even in the special cases that $K = 1$ **and** the "independent time-change" assumption is in force, the value of $\hat{\Theta}_t^{\mathrm{LFC}}$ is not constant (because the resulting solution depends upon both $y_t^{(1)}$ and $\left[\tilde{Y}_\bullet^{(1)}, \tilde{Y}_\bullet^{(1)}\right]_t$). Therefore, for variant (b) (i.e. when the aim is to minimise $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$), hedging strategy B would not reduce to a static buy-and-hold strategy (except when the "deterministic time-change" assumption is in force - in which case variant (b) amounts to the same as variant (a)). As already alluded to, this has practical implications since a dynamic position in LFC may incur significant transactions costs, in practice (see remark (3.1)).

5.3. **General remarks on the relative merits of variant** (a) **(minimising** $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$**) and variant** (b) **(minimising** $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$**).** We know that minimising $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ is the same as minimising $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ under the "deterministic time-change" assumption.

We saw in section (4) that, for hedging strategy A under the "common time-change" assumption, that when we minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$, it turns out that $Var^{\mathbb{Q}}_{t_0}[\epsilon_C(T)] = Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T), \epsilon_L(T)] = 0$ and, hence, $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)] = Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$.

Note that the terms $\left[\tilde{J}^{(j)}_{\ell,Y_\bullet}, \tilde{Y}^{(k)}_\bullet\right]_t$, for $\ell = 1,2,3,4$, will vanish under the "independent time-change" assumption. Furthermore, the terms $\left[\tilde{Y}^{(k)}_\bullet, \tilde{Y}^{(j)}_\bullet\right]_t$, for all $k, j = 1, 2, \ldots, K$, and the terms $\left[\tilde{J}^{(j)}_{\ell,Y_\bullet}, \tilde{Y}^{(k)}_\bullet\right]_t$, for $\ell = 1,2,3,4$, vanish under the "deterministic time-change" assumption. These terms will also always vanish at time t_0. This, perhaps, provides some intuition that the additional terms in equation (26) compared to equation (24) and more generally the additional terms appearing when one seeks to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ compared to when one seeks to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ may, in practice, typically be quite small - particularly for short-dated instruments (say, less than one year which is usually the case in practice for variance swaps).

This leads us to the following conjecture:

Conjecture 5.1. In practice, with data typical for equity options markets, if we select the values of ϕ_t (and when appropriate Θ^{LFC}_t and Θ^{SKS}_t) based on seeking to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ (rather than $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$), we will obtain values of $\hat{\phi}_t$ (and when appropriate $\hat{\Theta}^{\text{LFC}}_t$ and $\hat{\Theta}^{\text{SKS}}_t$) which are very close to those we would have obtained if we had sought to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$. Furthermore, the residual values of $Var^{\mathbb{Q}}_{t_0}[\epsilon_C(T)]$ and of $Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T), \epsilon_L(T)]$ will be very small and hence the residual values of $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ will be very close to the minimum values of $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$.

Of course, by seeking to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$, rather than $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$, we give up the potential for some further optimisation. However, the practical benefits are that minimising $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$:

- 1. Is much simpler and is potentially much more robust to model mis-specification.
- 2. Leads to static buy-and-hold positions in LFC (and when appropriate SKS) under the "common time-change" assumption which gives practical robustness to the possible presence (see remark (3.1)) of transactions costs.

We will put conjecture (5.1) to the test in the next section where we will provide numerical results for hedging variance swaps under realistic market dynamics where the hedges are determined by seeking to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$. We will demonstrate that the resulting residual values of $Var^{\mathbb{Q}}_{t_0}[\epsilon_C(T)]$ and of $Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T), \epsilon_L(T)]$ are, in practice, very small (they are equal to zero, as already noted, for hedging strategy A under the "common time-change" assumption) and that the residual values of $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ will be very close to the minimum values of $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ and, hence, that, in practice, selecting the values of ϕ_t (and when appropriate Θ^{LFC}_t and Θ^{SKS}_t) based on seeking to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ (rather than $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$) essentially works just as well whilst providing simplifications that are of practical benefit.

6. Numerical results

We now present some numerical results which illustrate our analysis in sections (3), (4) and (5).

Throughout this section, we assume a "common time-change" and use the solutions of equation (23) (i.e. variant (a)) to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$ (but we always report the residual value of $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ since it is this that is a measure of the total residual hedging error). With a "common time-change", we will always have constant values of $\hat{\phi}_t$ and (when appropriate) $\hat{\Theta}^{\mathrm{LFC}}_t$ and $\hat{\Theta}^{\mathrm{SKS}}_t$. A benefit of this is that we will always be able to compute $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ analytically using equations (16) and (17).

We consider a continuously monitored variance swap with maturity $T = 0.5$. Furthermore, in order to focus on the essentials, we assume zero interest-rates and zero dividend yield on the underlying stock.

This section is divided into five sub-sections. In the first sub-section, we assume a "deterministic time-change". In the second, we generalise by allowing for stochastic time-changes (and for non-zero covariances between the underlying returns processes $X^{(k)}_t$ and the stochastic time-changes $Y^{(k)}_t$). In the third, we also have stochastic time-changes and we use parameters obtained for time-changed CGMY processes from market data that have appeared in the extant literature. The fourth considers some robustness tests that we performed while the fifth summarises the first four.

6.1. **Numerical results with a "deterministic time-change".** We assume both a "common time-change" and a "deterministic time-change" and scale the time-change so that $\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}] = Y_T - Y_{t_0} = T = 0.5$.

We consider three different groupings of underlying processes for the stock with the results being reported in tables 1, 2 and 3 respectively.

For each grouping, we consider six different combinations of parameters and five different hedging strategies. The first hedging strategy (which is just for comparison purposes) uses the $2+2$ log-contract replication approach. The second hedging strategy (which is also just for comparison purposes) uses the approach of Schoutens (2005) and sets $\phi_t = 2$, $\Theta^{\mathrm{LFC}}_t = 2$, $\Theta^{\mathrm{SKS}}_t = 1/3$, for all t. This approach is labelled $2 + 2 + 1/3$ in the tables. The third, fourth and fifth hedging strategies correspond to hedging strategies A, B and C respectively.

In the first group (in table 1), we consider six combinations of (up to) three compound Poisson processes with fixed jump sizes and a single Brownian motion. In some cases, we set one or more of the intensity rates (labelled λ_1, λ_2, λ_3 in table 1) and/or the volatility of the Brownian motion (labelled "Vol" in table 1) equal to zero and thus effectively removing that process. The parameter combinations for the first group (labelled "params" 1 to 6 in table 1) were chosen by us. They are designed to be very approximately representative of the sorts of values that might be obtained if we had performed a calibration to the market prices of vanilla options in typical equity markets.

In the second group (in table 2), we consider, for six different combinations of parameters, a generalised CGMY process (generalised in the sense that we sometimes allow different C_{Up}, C_{Down} and Y_{Up}, Y_{Down} parameters - see Carr et al. (2002), (2003)) but with no diffusion component. We chose the parameters (labelled "params" 7 to 12 in table 2) as follows: Params 7 and 8 are parameter estimates from calibrations to the market prices of vanilla options on the S & P 500 stock index (for September 2000 and March 2000 respectively) and are quoted from Carr and Lee (2009). Params 9, 10, 11 and 12 are parameter sets which we made up to illustrate our results. Params 9 implies negatively skewed stock returns (since $M > G$). In params 10 we reverse the "up" and "down" parameters compared to params 9 so stock returns would be (for the puposes of illustration) positively skewed with these parameters. In params 11 and in params 12, we have symmetric Lévy measures (since $M = G$) with the difference being that, with params 12, we typically have much larger jumps (since the values of $M = G$ are one quarter those in params 12).

In the third group (in table 3), we consider a generalised CGMY process with the same values of M, G, Y_{Up} and Y_{Down} as in the second group but now we also have a diffusion component whose volatility is 0.1 (labelled "Vol" in table 3). The parameters are labelled params 13 to params 18.

For all three groups of parameters, we performed a scaling on C_{Up} and C_{Down} so that the (annualised) variance swap rate expressed as a volatility is always 0.25. For each set of parameters, we also give, in the relevant table, the price of (the floating leg of) a continuously monitored skewness swap and the value of Q_X.

For all 90 combinations of process type, parameters and hedging strategy we give the variance $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ (in **bold**) of the hedging strategy ($Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ is computed using equation (17)) (all values of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ are multiplied by 1,000,000 in table 1 and by 100 in tables 2 and 3 to improve readability) as well as the values of ϕ_t, Θ_t^{LFC}, Θ_t^{SKS} (with the hat symbol when we optimised over that portfolio weight). The results are displayed in tables 1, 2 and 3.

We make the following comments about the results:

In every case, hedging strategy C performs the best in the sense that $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ is smallest. Overall, hedging strategy B clearly performs the next best.

Overall, the standard $2+2$ log-contract replication approach performs the worst of the five hedging strategies. The $2+2+1/3$ approach generally performs much better than the standard 2+2 log-contract replication approach. It performs worse in two out of eighteen cases (params 8 and params 14) (because the hedge of $\Theta_t^{\text{SKS}} = 1/3$ is very far from the optimal values of 0.0944068 and 0.0956229). Hedging strategy A always significantly outperforms the standard $2+2$ log-contract replication approach although, comparing it with hedging strategy B, it is generally far from optimal. Of course, hedging strategy B always outperforms both the standard $2+2$ log-contract replication approach and hedging strategy A.

Observing table 1, we see that hedging strategy B gives perfect hedges for params 1 and 2 (in line with our analysis in special cases B1 and B2) while hedging strategy C also gives perfect hedges for params 3 and 4 (in line with our analysis in special cases C1 and C2). Clearly, hedging strategies B and C perform extremely well for params 1 to 6. However, we caution that this is essentially true almost by design of the parameters used. In the more realistic cases (params 7 to 18) where there are a continuum of possible jump amplitudes, the relative advantage of hedging strategies B and C, compared to the standard $2 + 2$ log-contract replication approach, is much reduced.

From the tables, one can see that, when Q_X is very far from two, then the optimal values $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$ are also very far from two. In particular, when Q_X is much greater than two (which implies that stock price returns, under $\mathbb{Q}$, are negatively skewed and that the value of the price of the floating leg of a skewness swap is large and negative), then the optimal values $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$ are much greater than two. In fact, in table 2, for params 8, the optimal values $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$ for hedging strategy B are actually greater than 4 which is to be compared with the values of 2 used in the standard $2 + 2$ log-contract replication approach - in other words, the standard $2 + 2$ log-contract replication approach uses hedges which are more than 100 % different from the optimal hedges. This is noteworthy as the parameters for this case were based on a calibration to market prices of vanilla options on the S & P 500 stock index (quoted from Carr and Lee (2009)).

The results for params 11, params 12, params 17 and params 18 show that, if one uses CGMY data which implies symmetric Lévy measures, then increasing the typical size of the jumps (i.e. making the parameters M and G smaller) causes the variance $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ of the hedging error to increase.

The key observation is that it is clear that the standard $2 + 2$ log-contract replication approach is very, very far from optimal in the presence of jumps (especially asymmetric jumps) - and, of course, in practice, equity markets exhibit (asymmetric) jumps.

6.2. **Numerical results with stochastic time-changes.** In this sub-section, we use a stochastic time-change. Specifically, the activity rate follows a Heston (1993) process of the form:

$$dy_t = \kappa(\eta - y_t)dt + \lambda y_t^{1/2} dz_t, \quad y_{t_0} \equiv y_0, \text{ with } y_0 > 0, \tag{27}$$

where z_t is a standard Brownian motion and $\kappa > 0$, $\eta > 0$ and $\lambda \geq 0$ are constants. The process that we time-change is exactly as in table 3 of the previous sub-section i.e. a generalised CGMY process with a diffusion component and with the same parameter values as before. We allow the Brownian motion z_t driving the activity rate y_t to have correlation ρ with the diffusion component of the generalised CGMY process. We consider three different values of ρ, namely $\rho = -0.99$, $\rho = 0$ and $\rho = 0.99$. We chose these three values simply for illustration as they will give us an opportunity to see how sensitive our analysis is to non-zero covariance between the underlying returns process (i.e. the $X_t^{(k)}$) and the stochastic time-change process (i.e. Y_t) over a range which is close to the maximum possible range of ρ from -1 to 1. We now discuss the choices of κ and λ

that we will use in the numerical examples in this sub-section. Our choices are based on table 7.3 of Schoutens (2003) and on table 9.3 on page 374 of Carr et al. (2003) where calibrations to market prices of vanilla options on the S & P 500 stock index were performed on different dates for a CGMY process time-changed by a Heston (1993) activity rate. Our aim is to try to lend weight to conjecture (5.1) in section (5). Using standard results for Heston (1993) activity rate dynamics, it is straighforward to see that $Var^{\mathbb{Q}}_{t_0}[\epsilon_C(T)]$ and $Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T), \epsilon_L(T)]$ will be proportional to λ^2/κ^3 and to λ/κ^2 respectively. To be as conservative as possible in testing conjecture (5.1), we should find values of λ^2/κ^3 and/or λ/κ^2 that are as large as possible. Therefore, we chose $\lambda = 1.3612$ and $\kappa = 0.3881$ (see the first row of table 7.3 of Schoutens (2003)) because out of all the calibrations referred to above in Schoutens (2003) and Carr et al. (2003), these choices gave by far the largest value of λ^2/κ^3 (and, as it happens, also the largest value of λ/κ^2).

We set $y_0 = 1$ and $\eta = 1$. We chose the values of ϕ_t (and when appropriate Θ^{LFC}_t and Θ^{SKS}_t) based on our analysis in section (4) (variant (a)) where we sought to minimise $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$.

Note that all results are values of $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ (and not of $Var^{\mathbb{Q}}_{t_0}[\epsilon_L(T)]$) and are reported in table 4. We have not included the values of ϕ_t, Θ^{LFC}_t, etc, in table 4 since they are, of course, the same as in table 3. Note also that the price of the floating leg of a continuously monitored skewness swap, the value of Q_X and the (annualised) variance swap rate expressed as a volatility are unchanged compared to their values in table 3 because these quantities only depend on $\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}] = T = 0.5$ and not on the values of ρ or λ.

For convenience, we also repeat, in table 4, the results of table 3 where we used a "deterministic time-change". These latter results are labelled "$\lambda = 0$". In table 4, we list the values of $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ for the four parameters combinations ($\lambda = 0$, $\{\lambda = 1.3612, \rho = 0\}$, $\{\lambda = 1.3612, \rho = -0.99\}$ and $\{\lambda = 1.3612, \rho = 0.99\}$) for the same six different combinations of generalised CGMY parameters as in table 3 and for each of the same five hedging strategies.

Note that, as already discussed in section (4), for hedging strategy A, there are no differences in the values of $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ compared to those in table 3 (because we have a "common time-change" and hence $Var^{\mathbb{Q}}_{t_0}[\epsilon_C(T)] = 0$ and $Covar^{\mathbb{Q}}_{t_0}[\epsilon_C(T), \epsilon_L(T)] = 0$). There are also no differences in the values of $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ for the cases $\{\lambda = 1.3612, \rho = 0\}$, $\{\lambda = 1.3612, \rho = -0.99\}$ and $\{\lambda = 1.3612, \rho = 0.99\}$, for the standard $2+2$ log-contract replication approach and the $2+2+1/3$ approach. There are differences in the values of $Var^{\mathbb{Q}}_{t_0}[\epsilon(T)]$ for hedging strategies B and C but they are very, very small.

As in the previous sub-section, hedging strategies A, B and C perform much better than the standard $2+2$ log-contract replication approach. In fact, the relative orderings of the performances of the five different hedging strategies are scarcely changed compared to those for the case of a "deterministic time-change" in table 3. In particular, hedging strategy C clearly outperforms hedging strategy B which, in turn, clearly outperforms hedging strategy A which, in turn, clearly outperforms the standard $2 + 2$ log-contract replication approach.

This lends considerable weight in support of conjecture (5.1) in section (5), namely, that if we select the values of ϕ_t (and when appropriate Θ_t^{LFC} and Θ_t^{SKS}) based on seeking to minimise $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$ (rather than $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$), the residual values of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ will be very close to the minimum values of $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$. Now, of course, different empirical data could lead to a different conclusion. The aim of this sub-section is most certainly not to perform an exhaustive empirical study - our results are simply designed to be illustrative. However, we repeat that our values of λ and κ were chosen to be as conservative as possible out of those listed in Carr et al. (2003) and in Schoutens (2003).

6.3. **More numerical results with stochastic time-changes based on Schoutens (2003) and Carr et al. (2003).** In this sub-section, we give more numerical results with stochastic time-changes. As we have already indicated, we are not aiming to perform an exhaustive empirical study and, of course, any study runs the risk of "data-mining". Therefore, in this sub-section, we take parameter values (obtained from calibrations to market prices of vanilla options on the S & P 500 stock index) for time-changed CGMY processes (with no diffusion component) directly, and without modification, out of table 7.3 of Schoutens (2003) and out of table 9.3 of Carr et al. (2003). The first five sets of parameters, labelled params 19 to params 23, use activity rates which follow a Heston (1993) process. The sixth set of parameters, params 24, uses an activity rate which follows a Gamma-OU process (see Barndorff-Nielsen and Shephard (2001)) of the form: $dy_t = -\lambda y_t dt + dZ_{\lambda t},\ y_{t_0} \equiv y_0$, with $y_0 > 0$, where Z_t is a compound Poisson process with intensity rate a, $a > 0$, and with exponentially distributed jumps with mean $1/b$, $b > 0$, and where $\lambda > 0$ is a constant. Finally, we use a seventh set of parameters, params 25, obtained by calibrating a CGMY process, time-changed using a Heston (1993) activity rate, to the market prices of options on the Nikkei-225 stock index as used in Crosby and Davis (2010) and as mentioned in section (1). The parameter values are listed in table 5 as are the values of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$. The column labelled "Var swap rate (as vol)" gives the (annualised) variance swap rate expressed as a volatility.

In broad terms, the results give further support to the conclusions of the previous two sub-sections. In particular, hedging strategy C clearly outperforms hedging strategy B which, in turn, clearly outperforms hedging strategy A which, in turn, clearly outperforms the standard 2 + 2 log-contract replication approach.

We saw in sub-section (4.1) that, under the "common time-change" assumption, with hedging strategy A, because $\Theta_t^{\mathrm{LFC}} = Q_X$, we have $Var_{t_0}^{\mathbb{Q}}[\epsilon_C(T)] = Covar_{t_0}^{\mathbb{Q}}[\epsilon_C(T), \epsilon_L(T)] = 0$ and hence $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)] = Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$. On the other hand, hedging strategy A gives up the possibility of further optimisation by fixing $\Theta_t^{\mathrm{LFC}} = Q_X$. Hedging strategy B does not result in $Var_{t_0}^{\mathbb{Q}}[\epsilon_C(T)] = Covar_{t_0}^{\mathbb{Q}}[\epsilon_C(T), \epsilon_L(T)] = 0$ (except under the "deterministic time-change" assumption) but further optimises over the choice of Θ_t^{LFC}. We can see from table 5 that it is the latter which seems to be more important than the former because hedging strategy B results in values of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ which are approximately one-third to one-half of those resulting from hedging strategy A.

The first parameter set, params 19, implies a particularly highly negatively skewed risk-neutral distribution (Q_X is much larger than two and the price of the floating leg of a skewness swap is large and negative). It results in optimal values of $\hat{\phi}_t$ (= 10.8956531) and $\hat{\Theta}_t^{\mathrm{LFC}}$ (= 9.9700106) for hedging strategy B which are approximately **five times larger** than the corresponding values implicit within the standard $2+2$ log-contract replication approach. The $2+2+1/3$ approach works particularly badly (and, in fact, much worse than the standard $2+2$ log-contract replication approach) with this parameter set - essentially because the the hedge of $\Theta_t^{\mathrm{SKS}} = 1/3$ is very far from the optimal value of 0.0224811. The $2+2+1/3$ approach also performs worse than the standard $2+2$ log-contract replication approach for params 20 and params 21 - for essentially the same reason. This certainly cautions against the naive use of $\Theta_t^{\mathrm{SKS}} = 1/3$. By contrast, hedging strategy C, which optimises over the choice of Θ_t^{SKS} (and of Θ_t^{LFC} and ϕ_t), always outperforms all the other hedging strategies.

Table 5 shows that the optimal values of $\hat{\phi}_t$ (and when appropriate $\hat{\Theta}_t^{\mathrm{LFC}}$ and $\hat{\Theta}_t^{\mathrm{SKS}}$) are highly dependent upon the skewness (under $\mathbb{Q}$) of the Lévy process (as measured by the value of Q_X). However, even for params 24, where the Lévy process is least negatively skewed (because it has the value of Q_X closest to two), for hedging strategy B, the optimal values of $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$ are more than 23 % larger than the corresponding values implicit within the standard $2+2$ log-contract replication approach. This is quite a large difference - especially when one considers that variance swaps are usually considered to be simple "flow" derivatives rather than highly exotic derivatives.

6.4. **Robustness tests.** In this sub-section, we give the results of some robustness tests that we have performed. In practice, delta-hedges are discretely rebalanced and the payoffs of variance swaps are based on discretely monitored (usually daily) stock prices. We have thus far used "minimising variance under $\mathbb{Q}$" as our "criterion of optimality" but another possible criterion is "pricing and hedging to acceptability". We now consider how robust our results are to both these issues. We simulated, using Monte Carlo simulation, the stock price on a discrete time grid $0 \equiv t_0 < t_1 < \ldots < t_{j-1} < t_j < \ldots < t_N \equiv T$. For constants ϕ_{D} and $\Theta_{\mathrm{D}}^{\mathrm{LFC}}$ ("D" is for discrete), we compute:

$$\begin{aligned}
\mathrm{P\&L} \equiv \Big(\sum_{j=1}^{N}(\log(S(t_j)/S(t_{j-1})))^2\Big) \quad &+ \quad \Theta_{\mathrm{D}}^{\mathrm{LFC}} \log\Big(S(T)/\big(S(t_0)\exp(\int_{t_0}^{T}(r(s)-q(s))ds)\big)\Big) \\
&- \quad \phi_{\mathrm{D}}\Big(\sum_{j=1}^{N}\frac{(S(t_j)-S(t_{j-1}))}{S(t_{j-1})}\Big). \qquad (28)
\end{aligned}$$

We then solve for optimal values $\hat{\phi}_{\mathrm{DMV}}$ and $\hat{\Theta}_{\mathrm{DMV}}^{\mathrm{LFC}}$ ("MV" is for minimising variance) of ϕ_{D} and $\Theta_{\mathrm{D}}^{\mathrm{LFC}}$ which minimise $Var_{t_0}^{\mathbb{Q}}[\mathrm{P\&L}]$.

Equation (28) is a discrete-time analogue to equations (11), (12) and (13) (with $\Theta_t^{\mathrm{VS}} \equiv 1$ and $\Theta_t^{\mathrm{SKS}} \equiv 0$ - for brevity, we did not consider skewness swaps in this exercise). In principle, we could also have allowed for ϕ_{D} and $\hat{\phi}_{\mathrm{DMV}}$ to be different at each time point t_{j-1} for $j = 1, 2, \ldots, N$ but again, for brevity, we did not consider this and assuming a constant value $\hat{\phi}_{\mathrm{DMV}}$ seems, roughly speaking, more in line with our previous numerical examples.

With the same Monte Carlo paths, we also used the "pricing and hedging to acceptability" methodology of Cherny and Madan (2010). This methodology computes a lower price and an upper price which, for brevity, we will term the "bid price" and the "offer price" (and labelled as such in table 6 - see from the fourth to the tenth rows), by evaluating expected discounted payoffs via a concave distortion function. We used the distortion function termed MINMAXVAR by Cherny and Madan (2010) (see their equation (21)), of the form: $\Psi^{\lambda}(u) = 1 - (1 - u^{\frac{1}{1+\lambda}})^{(1+\lambda)}$. We considered three different values of λ, namely $\lambda = 0.25$, $\lambda = 0.5$ and $\lambda = 0.75$ - the second and third being suggested by Cherny and Madan (2010) as appropriate for options written on the S & P 500 stock index while the first provides a contrast. This methodology also computes optimal hedges for each of the bid and offer prices (for brevity, we must refer the reader to Cherny and Madan (2010) for all details). The optimal hedges are assumed to be constants and of an analogous form to those in equation (28). In the obvious fashion, these are labelled $\hat{\phi}_{\mathrm{A}}$ and $\hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}$ ("A" is for acceptability") for bid and offer prices and for the three different values of λ in table 6.

We considered discretely monitored variance swaps with maturity $T = 0.5$ (as in all our previous numerical examples) and with $N = 126$ (in equation (28)) which corresponds approximately to daily monitoring. The results, using $150,000$ Monte Carlo simulations, are reported in table 6.

Our Monte Carlo simulation is only able to simulate a CGMY process (Carr et al. (2003)) when the Y_{Up} and Y_{Down} parameters are both strictly less than one (the finite variation case). Therefore, we report our results for the sets of parameters, from our previous numerical examples, for which this is the case - namely params 9, 10, 11, 12 and 25. Recall that params 25 also allows for a stochastic time-change (with the activity rate following a Heston (1993) process) of the CGMY process.

We report the Monte Carlo prices of the variance swap and the standard error of the price estimate in the second and third rows. Crosby and Davis (2010) show, with a similar data set, that the prices of variance swaps with daily monitoring should be close to those of continuously monitored variance swaps (reported in the first row and computed via equation (9)) and our numerical results support this. We see that the optimal hedges $\hat{\phi}_{\mathrm{DMV}}$ and $\hat{\Theta}_{\mathrm{DMV}}^{\mathrm{LFC}}$ in the discretely monitored case (in the twelth and thirteenth rows) are also close to the values of $\hat{\phi}_t$ and $\hat{\Theta}_t^{\mathrm{LFC}}$ (in the tenth and eleventh rows) for hedging strategy B and variant (a) (previously reported in tables 2 and 5).

We also compute, using our Monte Carlo simulation, and report in table 6 (see from the 26^{th} to the 30^{th} rows), the 99^{th} percentile Value-at-Risk (VAR) for a hedged long position in one variance swap for the cases when $\{\phi_{\mathrm{D}} = 2, \Theta_{\mathrm{D}}^{\mathrm{LFC}} = 2\}$ (corresponding to a discretely monitored, discretely hedged version of the "standard $2 + 2$ log-contract replication" approach), when $\{\phi_{\mathrm{D}} = \hat{\phi}_{\mathrm{DMV}}, \Theta_{\mathrm{D}}^{\mathrm{LFC}} = \hat{\Theta}_{\mathrm{DMV}}^{\mathrm{LFC}}\}$ and then for the cases when the positions $\{\phi_{\mathrm{D}}, \Theta_{\mathrm{D}}^{\mathrm{LFC}}\}$ are those computed using the "pricing and hedging to acceptability" approach for the three different values of λ. We can see that in nearly all cases, the VAR is significantly reduced for all the other hedged positions compared to the case when $\{\phi_{\mathrm{D}} = 2, \Theta_{\mathrm{D}}^{\mathrm{LFC}} = 2\}$.

From table 6, we see that, whilst the optimal hedges do, obviously, depend upon the "criterion of optimality" chosen, the optimal hedges obtained from the "pricing and hedging to acceptability" approach (see from the 14^{th} to the 25^{th} rows), broadly speaking, support the the conclusions of the previous sub-sections. In particular when Q_X is much greater than two, then the optimal values $\hat{\phi}_{\mathrm{DMV}}$, $\hat{\Theta}_{\mathrm{DMV}}^{\mathrm{LFC}}$, $\hat{\phi}_{\mathrm{A}}$, $\hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}$ are much greater than two. This in turn gives us grounds to believe that our conclusions are reasonably robust to both (a) discrete hedging / discrete monitoring and (b) the chosen "criterion of optimality".

6.5. **Summary of numerical results.** The results of the previous three sub-sections certainly give considerable weight to support conjecture (5.1) in section (5). Therefore, if the aim is to give practical recommendations to traders of variance derivatives as to the most appropriate way of hedging, we recommend using variant (a) (i.e. the hedges obtained by minimising $Var_{t_0}^{\mathbb{Q}}[\epsilon_L(T)]$ as described in section (4)). The resulting hedges are, in practice, likely to be very close (and indeed identical in some special cases of interest as we have already discussed in detail) to those that would be obtained using variant (b) (i.e. the hedges obtained by minimising $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$). Moreover, variant (a) generates simpler equations and under the "common time-change" assumption the resulting hedges have greater robustness (see remark (3.1)) to the possible presence, in practice, of transactions costs (because the hedges imply a static buy-and-hold position in log-forward-contracts and in skewness swaps) and greater robustness to model mis-specification (because the hedges do not depend upon the time-change process in any way). Our hedges are, of course, highly dependent upon the parameters of the Lévy processes - especially the skewness. Carr and Lee (2009) show (see also remark (2.9)) that, under the "common time-change" assumption, the price of a variance swap divided by the price of a log-forward-contract does not depend upon the time-change process in any way but is highly dependent upon the parameters of the Lévy processes - especially the skewness. Of course, vanilla options, to the market prices of which these time-changed Lévy process models are calibrated, are sensitive to both the skewness of the Lévy process(es) and to the covariance between the Lévy process(es) and the time-change. This dual-dependence makes it a challenge to distinguish how much of the skewness seen in implied volatility surfaces should be explained by the skewness of the Lévy process(es) (which severely impacts the pricing and hedging of variance swaps) and how much should be explained by a non-zero covariance between the Lévy process(es) and the time-change (which has little or no impact on the pricing and hedging of variance swaps, under already stated assumptions). Despite substantial research in this area (Carr et al. (2003), Duffie et al. (2000), Barndorff-Nielsen and Shephard (2001)), there is still no clear answer but our results, as well as those of Carr and Lee (2009), certainly motivate additional empirical research in the future. In the absence of additional empirical research, we have used data obtained from widely-referenced sources (Schoutens (2003) and Carr et al. (2003)) that is perceived to be "typical" for the S & P 500 stock index. With this data, the optimal (or nearly optimal) hedges that we have derived are so far from those implicit within the standard $2+2$ log-contract replication approach that we feel that our results should have practical implications for traders of variance swaps.

Both of the "criterion of optimality" that we have used, model the dynamics under the risk-neutral measure $\mathbb{Q}$. Empirical research (see Broadie et al. (2007) and the references therein) suggests that, for equity index options, jumps are larger in magnitude (i.e. more negative) and/or more frequent under the risk-neutral measure $\mathbb{Q}$ than under the real-world measure $\mathbb{P}$. Hence, if the chosen "criterion of optimality" involved an objective function computed under the real-world measure $\mathbb{P}$ - such as, for example, the "No Good Deals" approach of Cochrane and Saa-Requejo (2000) - our conclusions may be significantly revised. This suggests an interesting topic for further research. We thank an anonymous referee for making this observation.

7. Conclusions

We have examined the optimal hedging of variance swaps and, en route, also considered skewness swaps. We have shown that, in the presence of jumps in the underlying stock (which, in practice, is pertinent to all equity markets), the standard log-contract replication approach of Neuberger (1990) and Dupire (1993) provides relatively poor hedges and, in cases of practical interest with parameters obtained from calibrations to the market prices of vanilla options on the S & P 500 stock index, may imply hedges which are significantly different from the optimal (or nearly optimal) hedging strategies we have developed. We have derived formulae which give optimal (or nearly optimal) hedges under very general dynamics which allow for multiple jump processes and stochastic volatility. Throughout our analysis, we have sought to emphasize practical issues. The essence of our methodology has a degree of robustness to model mis-specification and to the possible presence, in practice, of transactions costs. We have also demonstrated, by numerical examples, that it also has a degree of robustness to both (a) discrete hedging / discrete monitoring and (b) the chosen "criterion of optimality".

Table 1.

	λ_1	a_1	λ_2	a_2	λ_3	a_3	Vol	Skewness swap price	Q_X
params 1	1.00000000	-0.2	0	0	0	0	0.15	-0.00400	2.0846708
params 2	1.53186275	-0.2	0.76593137	0.04	0	0	0	-0.00610	2.1320914
params 3	0.98039216	-0.2	0.49019608	0.04	0	0	0.15	-0.00391	2.0825752
params 4	1.50240385	-0.2	0.75120192	0.04	0.75120192	-0.04	0	-0.00601	2.1299626
params 5	0.96153846	-0.2	0.48076923	0.04	0.48076923	-0.04	0.15	-0.00385	2.0812748
params 6	0.54086538	-0.2	0.27043269	0.04	0.27043269	-0.04	0.2	-0.00216	2.0449185

For all parameters, the (annualised) variance swap rate expressed as a volatility is 0.25. For all hedging strategies, $\Theta_t^{\mathrm{VS}} = 1$. **All values of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ in the table below are multiplied by 1,000,000 to improve readability.** $T = 0.5$.

	params 1	params 2	params 3	params 4	params 5	params 6
$2+2$						
ϕ_t	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2
Θ_t^{SKS}	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**3.2219755**	**4.9358021**	**3.1589133**	**4.8410503**	**3.0982722**	**1.7427781**
$2+2+1/3$						
ϕ_t	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2
Θ_t^{SKS}	1/3	1/3	1/3	1/3	1/3	1/3
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0082141**	**0.0125829**	**0.0080531**	**0.0123410**	**0.0078982**	**0.0044428**
Hedge strategy A						
$\hat{\phi}_t$	2.0815517	2.1316674	2.0793692	2.1293857	2.0780750	2.0420725
Θ_t^{LFC}	2.0846708	2.1320914	2.0825752	2.1299626	2.0812748	2.0449185
Θ_t^{SKS}	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.1843912**	**0.0048679**	**0.1987352**	**0.0080840**	**0.2139058**	**0.5419420**
Hedge strategy B						
$\hat{\phi}_t$	2.1355255	2.1066839	2.1339678	2.1236177	2.1344956	2.1350182
$\hat{\Theta}_t^{\mathrm{LFC}}$	2.1355255	2.1093850	2.1341001	2.1247118	2.1345689	2.1350425
Θ_t^{SKS}	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0**	**0.0**	**0.0040225**	**0.0077286**	**0.0058494**	**0.0033147**
Hedge strategy C						
$\hat{\phi}_t$	2.1355255	2.1066839	1.9987087	1.9997334	1.9995097	1.9995006
$\hat{\Theta}_t^{\mathrm{LFC}}$	2.1355255	2.1093850	1.9987087	1.9997590	1.9995111	1.9995011
$\hat{\Theta}_t^{\mathrm{SKS}}$	0.0	0.0	0.3203351	0.3172952	0.3184241	0.3184699
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0**	**0.0**	**0.0**	**0.0**	**0.0000004**	**0.0000002**

Table 2.

	C_{Up}	C_{Down}	G	M	Y_{Up}	Y_{Down}	Vol	Skewness swap price	Q_X
params 7	0.60283195	0.04075144	1.64	16.9	-2.9	1.54	0	-0.00876	2.1675629
params 8	0.10998598	0.03170896	0.697	22	-3.65	1.45	0	-0.02466	2.4271496
params 9	0.08888068	0.60125165	3.34	14.64	0.165	0.165	0	-0.01696	2.3517572
params 10	0.60125165	0.08888068	14.64	3.34	0.165	0.165	0	0.01696	1.6245175
params 11	10.8377161	10.8377161	22.56	22.56	0.14	0.14	0	0	1.9982574
params 12	0.82244372	0.82244372	5.64	5.64	0.14	0.14	0	0	1.9719659

For all parameters, the (annualised) variance swap rate expressed as a volatility is 0.25. For all hedging strategies, $\Theta_t^{\text{VS}} = 1$. **All values of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ in the table below are multiplied by 100 to improve readability.** $T = 0.5$.

	params 7	params 8	params 9	params 10	params 11	params 12
$2+2$						
ϕ_t	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2
Θ_t^{SKS}	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.1003116**	**1.7636830**	**0.1296205**	**0.8684322**	**0.0001355**	**0.0422403**
$2+2+1/3$						
ϕ_t	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2
Θ_t^{SKS}	1/3	1/3	1/3	1/3	1/3	1/3
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0493292**	**4.5939681**	**0.0275941**	**0.2144251**	**0.0000007**	**0.0034313**
Strategy A						
$\hat{\phi}_t$	2.1395386	2.3243141	2.3158950	1.5145212	1.9947582	1.9120745
Θ_t^{LFC}	2.1675629	2.4271496	2.3517572	1.6245175	1.9982574	1.9719659
Θ_t^{SKS}	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0747423**	**1.2868488**	**0.0540592**	**0.2835614**	**0.0000966**	**0.0280183**
Strategy B						
$\hat{\phi}_t$	3.0508893	4.4583264	3.1344999	0.9105117	1.9879848	1.8068255
$\hat{\Theta}_t^{\text{LFC}}$	2.9968102	4.1897424	3.0142373	0.7132527	1.9914543	1.8587165
Θ_t^{SKS}	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0286036**	**0.5441049**	**0.0187595**	**0.0614244**	**0.0000962**	**0.0262567**
Strategy C						
$\hat{\phi}_t$	2.2981238	2.9930569	2.2546368	1.3987597	1.9938378	1.8962149
$\hat{\Theta}_t^{\text{LFC}}$	2.2865967	2.9064230	2.2358833	1.2574362	1.9938275	1.8935415
$\hat{\Theta}_t^{\text{SKS}}$	0.1600923	0.0944068	0.1831590	0.2800165	0.3339012	0.3416304
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0011205**	**0.0425269**	**0.0003991**	**0.0583729**	**0.0000004**	**0.0017009**

Table 3.

	C_{Up}	C_{Down}	G	M	Y_{Up}	Y_{Down}	Vol	Skewness swap price	Q_X
params 13	0.50637884	0.03423121	1.64	16.9	-2.9	1.54	0.1	-0.00736	2.1388910
params 14	0.09238822	0.02663552	0.697	22	-3.65	1.45	0.1	-0.02071	2.3469497
params 15	0.07465977	0.50505138	3.34	14.64	0.165	0.165	0.1	-0.01425	2.2873888
params 16	0.50505138	0.07465977	14.64	3.34	0.165	0.165	0.1	0.01425	1.6748270
params 17	9.10368153	9.10368153	22.56	22.56	0.14	0.14	0.1	0	1.9985360
params 18	0.69085272	0.69085272	5.64	5.64	0.14	0.14	0.1	0	1.9763984

For all parameters, the (annualised) variance swap rate expressed as a volatility is 0.25. For all hedging strategies, $\Theta_t^{\text{VS}} = 1$. **All values of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ in the table below are multiplied by 100 to improve readability.** $T = 0.5$.

	params 13	params 14	params 15	params 16	params 17	params 18
$2+2$						
ϕ_t	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2
Θ_t^{SKS}	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0842618**	**1.4814937**	**0.1088812**	**0.7294831**	**0.0001138**	**0.0354819**
$2+2+1/3$						
ϕ_t	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2
Θ_t^{SKS}	1/3	1/3	1/3	1/3	1/3	1/3
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0414365**	**3.8589332**	**0.0231790**	**0.1801171**	**0.0000006**	**0.0028823**
Strategy A						
$\hat{\phi}_t$	2.1139029	2.2550110	2.2480887	1.5585123	1.9955928	1.9250328
Θ_t^{LFC}	2.1388910	2.3469497	2.2873888	1.6748270	1.9985360	1.9763984
Θ_t^{SKS}	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0658955**	**1.1445256**	**0.0524545**	**0.2667471**	**0.0000863**	**0.0251997**
Strategy B						
$\hat{\phi}_t$	3.0017860	4.2801042	2.9618934	0.9700460	1.9876175	1.8019377
$\hat{\Theta}_t^{\text{LFC}}$	2.9608813	4.0900095	2.8915453	0.8233291	1.9905313	1.8454083
Θ_t^{SKS}	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0251331**	**0.4825763**	**0.0199881**	**0.0660671**	**0.0000859**	**0.0231835**
Strategy C						
$\hat{\phi}_t$	2.2820842	2.9182501	2.2055491	1.8132336	1.9938378	1.8962027
$\hat{\Theta}_t^{\text{LFC}}$	2.2736164	2.8585924	2.1963724	1.7409378	1.9938297	1.8940907
$\hat{\Theta}_t^{\text{SKS}}$	0.1613416	0.0956229	0.1886903	0.5008981	0.3338428	0.3406762
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**0.0009900**	**0.0383068**	**0.0004213**	**0.0541418**	**0.0000003**	**0.0014316**

Table 4.

For all parameters, the (annualised) variance swap rate expressed as a volatility is 0.25. For all hedging strategies, $\Theta_t^{\mathrm{VS}} = 1$. **All values in the table below are values of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ and are multiplied by 100 to improve readability.** $T = 0.5$.

	params 13	params 14	params 15	params 16	params 17	params 18
$2+2$						
$\lambda = 0$	**0.0842618**	**1.4814937**	**0.1088812**	**0.7294831**	**0.0001138**	**0.0354819**
$\rho = 0$	**0.0843720**	**1.4820649**	**0.1092938**	**0.7304684**	**0.0001138**	**0.0354856**
$\rho = -0.99$	**0.0843720**	**1.4820649**	**0.1092938**	**0.7304684**	**0.0001138**	**0.0354856**
$\rho = 0.99$	**0.0843720**	**1.4820649**	**0.1092938**	**0.7304684**	**0.0001138**	**0.0354856**
$2+2+1/3$						
$\lambda = 0$	**0.0414365**	**3.8589332**	**0.0231790**	**0.1801171**	**0.0000006**	**0.0028823**
$\rho = 0$	**0.0414414**	**3.8590729**	**0.0231972**	**0.1801635**	**0.0000006**	**0.0028860**
$\rho = -0.99$	**0.0414414**	**3.8590729**	**0.0231972**	**0.1801635**	**0.0000006**	**0.0028860**
$\rho = 0.99$	**0.0414414**	**3.8590729**	**0.0231972**	**0.1801635**	**0.0000006**	**0.0028860**
Strategy A						
$\lambda = 0$	**0.0658955**	**1.1445256**	**0.0524545**	**0.2667471**	**0.0000863**	**0.0251997**
$\rho = 0$	**0.0658955**	**1.1445256**	**0.0524545**	**0.2667471**	**0.0000863**	**0.0251997**
$\rho = -0.99$	**0.0658955**	**1.1445256**	**0.0524545**	**0.2667471**	**0.0000863**	**0.0251997**
$\rho = 0.99$	**0.0658955**	**1.1445256**	**0.0524545**	**0.2667471**	**0.0000863**	**0.0251997**
Strategy B						
$\lambda = 0$	**0.0251331**	**0.4825763**	**0.0199881**	**0.0660671**	**0.0000859**	**0.0231835**
$\rho = 0$	**0.0289937**	**0.4969947**	**0.0218116**	**0.0728237**	**0.0000863**	**0.0232983**
$\rho = -0.99$	**0.0276263**	**0.4847143**	**0.0201954**	**0.0793120**	**0.0000853**	**0.0230477**
$\rho = 0.99$	**0.0303611**	**0.5092751**	**0.0234278**	**0.0663355**	**0.0000873**	**0.0235489**
Strategy C						
$\lambda = 0$	**0.0009900**	**0.0383068**	**0.0004213**	**0.0541418**	**0.0000003**	**0.0014316**
$\rho = 0$	**0.0012566**	**0.0403764**	**0.0004772**	**0.0550748**	**0.0000004**	**0.0014769**
$\rho = -0.99$	**0.0011822**	**0.0389163**	**0.0004403**	**0.0562629**	**0.0000004**	**0.0014846**
$\rho = 0.99$	**0.0013310**	**0.0418366**	**0.0005142**	**0.0538867**	**0.0000004**	**0.0014693**

Table 5.

	C_{Up}	C_{Down}	G	M	Y_{Up}	Y_{Down}	Vol	Skewness swap price	Q_X
params 19	0.00740000	0.00740000	0.1025	11.394	1.6765	1.6765	0	-0.06977	2.7294158
params 20	0.16350000	0.04713705	0.6965	21.97	-3.65	1.45	0	-0.01272	2.4274086
params 21	0.35870000	0.01886762	0.4231	24.64	-4.51	1.67	0	-0.01419	2.3727413
params 22	0.40410000	0.02731716	1.64	16.91	-2.9	1.54	0	-0.00385	2.1675632
params 23	2.04400000	0.17476200	3.68	52.86	-2.12	1.22	0	-0.01054	2.1349535
params 24	0.04150000	0.04150000	3.9134	30.6322	1.3664	1.3664	0	-0.00182	2.0769284
params 25	1.69755908	1.69755908	6.647	78.61	0.2064	0.2064	0	-0.00772	2.1748812

Table 5 continued.

	Var swap rate (as vol)	λ	κ	η	y_0	ρ	Var swap price
params 19	0.232270	1.3612	0.3881	1.4012	1	0	0.0269747
params 20	0.179512	0.00022	8.51	0.1497	1	0	0.0161122
params 21	0.190740	0.0006	6.65	0.3469	1	0	0.0181909
params 22	0.165670	2.78E-05	4.85	0.4474	1	0	0.0137233
params 23	0.315297	1.7	15.91	1.3700	1	0	0.0497062
params 24	0.172255	0.8826	$a = 0.5945$	$b = 0.8524$	1	0	0.0148359
params 25	0.240812	1.8105	0.6578	1.5764	1	0	0.0289952

For all hedging strategies, $\Theta_t^{\text{VS}} = 1$. **All values of $Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$ in the table below are multiplied by 100 to improve readability.** $T = 0.5$.

	params 19	params 20	params 21	params 22	params 23	params 24	params 25
$2+2$							
ϕ_t	2	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2	2
Θ_t^{SKS}	0	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**69.47894**	**0.9111811**	**1.9158518**	**0.0440515**	**0.0252706**	**0.0032939**	**0.0102416**
$2+2+1/3$							
ϕ_t	2	2	2	2	2	2	2
Θ_t^{LFC}	2	2	2	2	2	2	2
Θ_t^{SKS}	1/3	1/3	1/3	1/3	1/3	1/3	1/3
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**6188.486**	**2.3766379**	**11.4521946**	**0.0216628**	**0.0030044**	**0.0003268**	**0.0005453**
Strategy A							
$\hat{\phi}_t$	2.4383574	2.3244859	2.2640247	2.1395390	2.1218021	2.0679356	2.1641369
Θ_t^{LFC}	2.7294158	2.4274086	2.3727413	2.1675632	2.1349535	2.0769284	2.1748812
Θ_t^{SKS}	0	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**62.97082**	**0.6648498**	**1.5885852**	**0.0328228**	**0.0156676**	**0.0024078**	**0.0041346**
Strategy B							
$\hat{\phi}_t$	10.8956531	4.4599057	5.0370276	3.0508929	2.6186136	2.4716678	2.5594479
$\hat{\Theta}_t^{\text{LFC}}$	9.9700106	4.1910341	4.7686888	2.9968132	2.5907777	2.4615637	2.5267807
Θ_t^{SKS}	0	0	0	0	0	0	0
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**33.61962**	**0.2811378**	**0.7166531**	**0.0125611**	**0.0056145**	**0.0010554**	**0.0021402**
Strategy C							
$\hat{\phi}_t$	7.1416567	2.9939131	3.4025189	2.2981263	2.1161261	2.0762521	2.0831422
$\hat{\Theta}_t^{\text{LFC}}$	6.6737900	2.9071631	3.3005068	2.2865989	2.1123662	2.0753046	2.0797957
$\hat{\Theta}_t^{\text{SKS}}$	0.0224811	0.0943611	0.0709055	0.1600920	0.2139766	0.2270833	0.2349488
$Var_{t_0}^{\mathbb{Q}}[\epsilon(T)]$	**6.12475**	**0.0219838**	**0.0774067**	**0.0004920**	**0.0000903**	**0.0000169**	**0.0000221**

Table 6.

	params 9	params 10	params 11	params 12	params 25
Var swap price (anal)	0.0312500	0.0312500	0.0312500	0.0312500	0.0289952
Var swap price (MC)	0.0314860	0.0309568	0.0312157	0.0310877	0.0291106
Standard error	0.0003252	0.0003072	0.0000477	0.0001872	0.0001590
Bid price $\lambda = 0.25$	0.0297194	0.0265521	0.0308601	0.0275996	0.0283367
Offer price $\lambda = 0.25$	0.0352066	0.0331717	0.0315674	0.0344164	0.0306272
Bid price $\lambda = 0.5$	0.0286490	0.0218774	0.0304760	0.0238521	0.0279574
Offer price $\lambda = 0.5$	0.0411256	0.0343744	0.0319382	0.0379767	0.0330430
Bid price $\lambda = 0.75$	0.0275657	0.0196013	0.0300212	0.0202451	0.0277353
Offer price $\lambda = 0.75$	0.0483797	0.0354357	0.0323684	0.0416845	0.0363403
Analytical					
$\hat{\phi}_t$	3.1344999	0.9105117	1.9879848	1.8068255	2.5594479
$\hat{\Theta}_t^{\mathrm{LFC}}$	3.0142373	0.7132527	1.9914543	1.8587165	2.5267807
Monte Carlo					
$\hat{\phi}_{\mathrm{DMV}}$	3.2145844	1.0088239	1.9878759	1.8171757	2.5075201
$\hat{\Theta}_{\mathrm{DMV}}^{\mathrm{LFC}}$	3.0809348	0.8469218	1.9916039	1.8701114	2.4812933
Hedging to acceptability					
Bid $\hat{\phi}_{\mathrm{A}}$ $\lambda = 0.25$	2.4685721	1.1694190	1.9630708	1.7367498	2.2004639
Bid $\hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}$ $\lambda = 0.25$	2.4424330	1.0970614	1.9657680	1.7529853	2.1970906
Offer $\hat{\phi}_{\mathrm{A}}$ $\lambda = 0.25$	2.8645302	1.5166750	2.0209049	2.0993438	2.3778078
Offer $\hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}$ $\lambda = 0.25$	2.8079582	1.4787809	2.0236749	2.1199459	2.3649492
Bid $\hat{\phi}_{\mathrm{A}}$ $\lambda = 0.5$	2.3578254	0.9129815	1.9353046	1.5291110	2.1576639
Bid $\hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}$ $\lambda = 0.5$	2.3421220	0.8303504	1.9378757	1.5404824	2.1559739
Offer $\hat{\phi}_{\mathrm{A}}$ $\lambda = 0.5$	3.2622818	1.5982663	2.0479313	2.2883086	2.5107230
Offer $\hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}$ $\lambda = 0.5$	3.1861281	1.5719362	2.0506504	2.3080368	2.4911953
Bid $\hat{\phi}_{\mathrm{A}}$ $\lambda = 0.75$	2.2566088	0.7513199	1.9058135	1.3263418	2.1303199
Bid $\hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}$ $\lambda = 0.75$	2.2493061	0.6716910	1.9082153	1.3344387	2.1294833
Offer $\hat{\phi}_{\mathrm{A}}$ $\lambda = 0.75$	3.6641479	1.6679601	2.0758198	2.4878403	2.7057963
Offer $\hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}$ $\lambda = 0.75$	3.5759660	1.6507373	2.0784538	2.5059862	2.6789849
99th percentile VAR					
VAR $\{\phi_{\mathrm{D}} = 2, \Theta_{\mathrm{D}}^{\mathrm{LFC}} = 2\}$	0.1532198	0.0691230	0.0656900	0.0907471	0.0933229
VAR $\{\hat{\phi}_{\mathrm{DMV}}, \hat{\Theta}_{\mathrm{DMV}}^{\mathrm{LFC}}\}$	0.0820607	0.1208635	0.0650907	0.0896958	0.0626494
VAR $\{\hat{\phi}_{\mathrm{A}}, \hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}\}$, $\lambda = 0.25$	0.0694683	0.0701591	0.0646238	0.0732686	0.0608514
VAR $\{\hat{\phi}_{\mathrm{A}}, \hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}\}$, $\lambda = 0.5$	0.0736461	0.0682568	0.0641386	0.0687674	0.0622854
VAR $\{\hat{\phi}_{\mathrm{A}}, \hat{\Theta}_{\mathrm{A}}^{\mathrm{LFC}}\}$, $\lambda = 0.75$	0.0776817	0.0674159	0.0638037	0.0710855	0.0646506

References:

Barndorff-Nielsen O. and N. Shephard (2001) "Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics" *Journal of the Royal Statistical Society* B Vol. 63 p167-241

Bates D. (1996) "Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options" *Review of Financial Studies* Vol. 9 p69-107

Broadie M., M. Chernov and M. Johannes (2007) "Model specification and risk premia: Evidence from futures options" *Journal of Finance* Vol. 62 No. 3 p1453-1490

Broadie M. and A. Jain (2008) "Pricing and Hedging Volatility Derivatives" *Journal of Derivatives* Vol. 15 No. 3 p7-24

Carr P., H. Geman, D. Madan and M. Yor (2002) "The fine structure of asset returns: An empirical investigation" *Journal of Business* Vol. 75 No. 2 p305-332

Carr P., H. Geman, D. Madan and M. Yor (2003) "Stochastic volatility for Lévy processes" *Mathematical Finance* Vol. 13 p345-382

Carr P. and R. Lee (2010) "Hedging variance options on continuous semimartingales" *Finance and Stochastics* Vol. 14 No. 2 p179-207

Carr P. and R. Lee (2009) "Pricing variance swaps on time-changed Lévy processes" Working paper - accepted for publication in revised form with additional empirical results as Carr et al. (2010)

Carr P., R. Lee and L. Wu (2010) "Variance swaps on time-changed Lévy processes" Accepted for publication in *Finance and Stochastics*

Carr P. and L. Wu (2007) "Stochastic skew in currency options" *Journal of Financial Economics* Vol. 86 p213-247

Cherny A. and D. Madan (2009) "New measures for performance evaluation" *Review of Financial Studies* Vol. 22 p2571-2606

Cherny A. and D. Madan (2010) "Markets as a counterparty: An introduction to conic finance" *International Journal of Theoretical and Applied Finance"* Vol. 13 No. 8 p1149-1177

Cochrane J. and J. Saa-Requejo (2000) "Beyond arbitrage: Good deal asset price bounds in incomplete markets" *Journal of Political Economy* Vol. 108 p79-119

Cont R. and P. Tankov (2004) "Financial modelling with jump processes" Chapman & Hall

Crosby J. and M. H. A. Davis (2010) "Variance derivatives: Pricing and convergence" Working paper

Demeterfi K., E. Derman, M. Kamal and J. Zou (1999) "More than you ever wanted to know about volatility swaps" *Journal of Derivatives* Vol. 6 No. 4 p9-32 (also a Goldman Sachs Quantitative Strategies note available on Emanuel Derman's website at http://www.ederman.com)

Duffie D., J. Pan and K. Singleton (2000) "Transform analysis and asset pricing for affine jump-diffusions" *Econometrica* Vol. 68 No. 6 p1343-1376

Dupire B. (1993) "Model art" *Risk* Vol. 6 No. 9 p118-124

Heston S. (1993) "A closed-form solution for options with stochastic volatility with applications to bond and currency options" *Review of Financial Studies* Vol. 6 p327-343

Kou S. (2002) "A jump-diffusion model for option pricing" *Management Science* Vol. 48 p1086-1101

Madan D. (2010) "Pricing and hedging basket options to prespecified levels of acceptability" *Quantitative Finance* Vol. 10 No. 6 p607-615

Merton R. (1976) "Option pricing when the underlying stock returns are discontinuous" *Journal of Financial Economics* Vol. 3 p115-144

Neuberger A. (1990) "Volatility trading" Working paper, London Business School

Neuberger A. (1994) "The Log Contract: A new instrument to hedge volatility" *Journal of Portfolio Management* Winter 1994 p74-80

Neuberger A. (1996) "The Log Contract and Other Power Contracts" in "The Handbook of Exotic Options" edited by I. Nelken p200-212

Schoutens W. (2003) "Lévy processes in finance: Pricing financial derivatives" Wiley, Chichester, United Kingdom

Schoutens W. (2005) "Moment swaps" *Quantitative Finance* Vol. 5 No. 6 p525-530