Convexity adjustments in inflation-linked derivatives

Dorje Brody, **John Crosby** and **Hongyun Li** *value several types of inflation-linked derivatives using a multi-factor version of the Hughston (1998) and Jarrow & Yildirim (2003) model. Expressions for the prices of zero-coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments are obtained in closed form by explicitly calculating the relevant convexity adjustments. These results are then applied to value limited price indexation swaps using Ryten's (2007) common factor representation methodology*

The market for inflation-linked derivatives has grown rapidly in recent years. Inflation is now regarded as an independent asset class. Actively traded inflation derivatives include standard zero-coupon inflation swaps, as well as more complicated products such as period-on-period inflation swaps (Mercurio, 2005), inflation caps (Mercurio, 2005), inflation swaptions (Kerkhof, 2005) and futures contracts written on inflation (Crosby, 2007).

Consider a standard zero-coupon inflation swap with maturity T_M, fixed rate K and notional amount N, which we enter into at time zero. Let X_t denote the spot consumer price index (CPI) at time t. The payout at time T_M of the standard zero-coupon inflation swap is $N(X_{T_M}/X_0 - 1) - N((1 + K)^{T_M} - 1)$. Notice that the time T_M at which the CPI is measured to specify the payout agrees with the time at which the payment takes place. While this is the usual situation, often in practice the payment is delayed until some later time $T_N \geq T_M$. This delay is not just the standard two-day spot settlement lag but can be a period of a few weeks, a few months or even several years. We will refer to such inflation swaps as 'inflation swaps with delayed payments'.

To see how such inflation swaps have an important economic rationale, consider a commercial property company. Suppose it has debt in the form of fixed-rate loans. It receives rents from its tenants that it wants to pay out as the inflation-linked leg of an inflation swap. It will receive fixed payments on the inflation swap, which is used to pay its fixed-rate debt. Often rents will remain constant for a period of five years before being reviewed. They will then be revised upwards to reflect inflation over those intervening five years. So, for example, suppose the commercial property company wanted to enter into an inflation swap trade, in which it paid inflation-linked cashflows and received fixed cashflows. The company wants to hedge the cashflows that it will receive from its tenants in years six, seven, eight, nine and 10. A suitable inflation swap trade would be a strip of five zero-coupon inflation swaps, where the payouts of the five zero-coupon swaps are (we write only the inflation-linked leg with unit notional) as follows: at the end of year six, the company pays $X_5/X_0 - 1$. At the end of year seven, it again pays $X_5/X_0 - 1$. Likewise, it pays $X_5/X_0 - 1$ at the end of years eight, nine and 10.

We see that these are zero-coupon inflation swaps with delayed payment, with the delay on the final strip being five years. Period-on-period swaps with delayed payments are also traded in the markets. We will provide formulas for both these types of inflation swap by calculating the relevant convexity adjustments. Note that the issue of delayed payments should not be confused with the issue of indexation lag. Indexation lag refers to the fact that the value of the CPI in the denominator of the inflation-linked term in the payout is, in fact, the CPI published (typically) a few weeks earlier, which, in turn, was calculated from consumer prices observed a few weeks before that. This is a different issue (although it would be possible to relate the two) and we refer the reader to Kerkhof (2005) and Li (2007).

Limited price indexation (LPI) swaps are a type of exotic inflation derivative and are very common in the UK owing to the rules by which UK pension funds are governed. We will see that the convexity adjustments required to value inflation swaps with delayed payments have a further application in the valuation of LPI swaps.

This article is structured as follows. First, we introduce the dynamics of nominal and real zero-coupon bond prices and the spot CPI. Then we state the convexity adjustments required to value zero-coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments. To our best knowledge, these results, in the context of a multi-factor Hughston (1998) and Jarrow & Yildirim (2003) model, have not appeared before, although some similar results (in the context of a two-factor Hull-White type model) are in Dodgson & Kainth (2006). These results are then applied to the valuation of LPI swaps, aided by the quasi-analytic methodology of Ryten (2007). A number of examples and comparisons are then given, and we finish with a brief concluding remark. Appendix A contains proofs of the convexity adjustment formulas.

Models for bond prices and the spot CPI

We model the market with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{0 \le t < \infty}$ generated by a multi-dimensional Brownian motion. The probability measure $\mathbb{Q}$ denotes the risk-neutral measure, and market prices and other information-providing processes are adapted to $\{\mathcal{F}_t\}$. Throughout the article, we assume the absence of arbitrage and the existence of a pricing kernel. These conditions ensure the existence of a unique pricing measure $\mathbb{Q}$. We let $\mathbb{E}_t[-]$ denote the expectation in $\mathbb{Q}$ conditional on $\{\mathcal{F}_t\}$.

We denote calendar time by t; time $t = 0$ will denote the initial time. Let $\{r_t^N\}$ and $\{r_t^R\}$ denote, respectively, the (continuously compounded) risk-free nominal and real short-rate processes. Let $\{P_{tT}^N\}$ and $\{P_{tT}^R\}$ denote, respectively, the price process of a nominal and real zero-coupon bond maturing at T. The spot CPI at time t is denoted by X_t.

A key observation for pricing inflation derivatives is that, for any times t and T_M, $t \le T_M$, we have (Hughston, 1998):

$$X_t P_{tT_M}^R = \mathbb{E}_t\left[X_{T_M}\exp\left(-\int_t^{T_M} r_s^N\,ds\right)\right] \tag{1}$$

This follows from the fact that the right-hand side of (1) is the price at time t of an index-linked bond, which pays the amount X_{T_M} at time T_M. Dividing it by X_t, we obtain the value in real terms of a bond that pays one unit of goods and services at time T_M. Mercurio (2005) uses this relation to value standard zero-coupon inflation swaps, and shows how, given the fixed rates quoted in the markets for these swaps, the term structure of real discount factors can be obtained.

Now we introduce the models for the dynamical equations satisfied by nominal zero-coupon bond prices, real zero-coupon bond prices and the spot CPI, within the multi-factor version of the Hughston (1998) and Jarrow & Yildirim (2003) model. These are given by:

$$\frac{dP_{tT}^N}{P_{tT}^N} = r_t^N\,dt + \sum_{k=1}^{K_N}\sigma_{ktT}^N\,dz_{kt}^N \tag{2}$$

$$\frac{dP_{tT}^R}{P_{tT}^R} = \left(r_t^R - \sum_{k=1}^{K_R}\rho_k^{RX}\sigma_t^X\sigma_{ktT}^R\right)dt + \sum_{k=1}^{K_R}\sigma_{ktT}^R\,dz_{kt}^R \tag{3}$$

and:

$$\frac{dX_t}{X_t} = \left(r_t^N - r_t^R\right)dt + \sigma_t^X\,dz_t^X \tag{4}$$

Here K_N and K_R are the number of Brownian motions driving nominal and real zero-coupon bond prices, respectively, $\{dz_{kt}^N\}_{k=1,\ldots,K_N}$, $\{dz_{kt}^R\}_{k=1,\ldots,K_R}$, and $\{dz_t^X\}$ denote standard $\mathbb{Q}$ Brownian increments. Furthermore, $\{\sigma_{ktT}^N\}_{k=1,\ldots,K_N}$ and $\{\sigma_{ktT}^R\}_{k=1,\ldots,K_R}$ are volatility terms, which are assumed to be deterministic, satisfying $\sigma_{kTT}^N = 0$, and $\{\sigma_t^X\}$ is the spot CPI volatility, which we also assume to be deterministic. We denote correlations (all assumed constant) by ρ with appropriate subscripts: $\mathrm{Corr}(dz_{jt}^N, dz_{kt}^N) = \rho_{jk}^{NN}dt$, $\mathrm{Corr}(dz_{jt}^R, dz_{kt}^R) = \rho_{jk}^{RR}dt$, $\mathrm{Corr}(dz_t^X, dz_{kt}^N) = \rho_k^{NX}dt$, $\mathrm{Corr}(dz_t^X, dz_{jt}^R) = \rho_j^{RX}dt$ and $\mathrm{Corr}(dz_{jt}^N, dz_{kt}^R) = \rho_{jk}^{NR}dt$.

LPI swaps

Here, we will provide a valuation formula for LPI swaps. Before discussing LPI swaps, we state two preliminary propositions, the proofs of which are in Appendix A. We will use them to value LPI swaps. They can also be used to value zero-coupon inflation swaps with delayed payments and period-on-period inflation swaps with delayed payments.

■ **Proposition 1.** Given the assumptions in the previous section, for any times t and T_N, $0 \le t \le T_M \le T_N$, the following relation holds:

$$\begin{aligned}&\mathbb{E}_t\left[X_{T_M}\exp\left(-\int_t^{T_N} r_s^N\,ds\right)\right]\\ &= X_t P_{tT_M}^R\frac{P_{tT_N}^N}{P_{tT_M}^N}\exp\left(\int_t^{T_M} C_s(T_M,T_N)\,ds\right)\end{aligned} \tag{5}$$

where:

$$\begin{aligned}&C_s(T_M,T_N)\\ &= \sum_{k=1}^{K_N}\left(\sigma_{ksT_N}^N - \sigma_{ksT_M}^N\right)\left(\sum_{j=1}^{K_R}\rho_{kj}^{NR}\sigma_{jsT_M}^R - \sum_{j=1}^{K_N}\rho_{kj}^{NN}\sigma_{jsT_M}^N\right)\\ &+\sum_{k=1}^{K_N}\left(\sigma_{ksT_N}^N - \sigma_{ksT_M}^N\right)\rho_k^{NX}\sigma_s^X\end{aligned} \tag{6}$$

When $T_M = T_N$ it is straightforward to verify that $C_s(T_M, T_N) = 0$, in which case equation (5) agrees with equation (1).

■ **Proposition 2.** Given the assumptions of the previous section, we have, for $0 \le t < T_{i-1} < T_i \le T_{N_i}$:

$$\begin{aligned}&\mathbb{E}_t\left[\frac{X_{T_i}}{X_{T_{i-1}}}\exp\left(-\int_t^{T_{N_i}} r_s^N\,ds\right)\right]\\ &= P_{tT_{i-1}}^N\frac{P_{tT_{N_i}}^N}{P_{tT_i}^N}\frac{P_{tT_i}^R}{P_{tT_{i-1}}^R}\exp\left(\int_{T_{i-1}}^{T_i} C_s\left(T_i,T_{N_i}\right)ds\right.\\ &\left.+\int_t^{T_{i-1}}\left[A_s\left(T_{i-1},T_i\right) + B_s\left(T_{i-1},T_i,T_{N_i}\right)\right]ds\right)\end{aligned} \tag{7}$$

where $C_s(T_i, T_{N_i})$ is given by (6) and where:

$$\begin{aligned}A_s\left(T_{i-1},T_i\right) &= \sum_{j=1}^{K_R}\left(\sigma_{jsT_i}^R - \sigma_{jsT_{i-1}}^R\right)\left(\sum_{k=1}^{K_N}\rho_{kj}^{NR}\sigma_{ksT_{i-1}}^N - \sum_{k=1}^{K_R}\rho_{kj}^{RR}\sigma_{ksT_{i-1}}^R\right)\\ &+\sum_{k=1}^{K_R}\left(\sigma_{ksT_{i-1}}^R - \sigma_{ksT_i}^R\right)\rho_k^{RX}\sigma_s^X\end{aligned} \tag{8}$$

and:

$$\begin{aligned}B_s\left(T_{i-1},T_i,T_{N_i}\right) &= \sum_{k=1}^{K_N}\sum_{j=1}^{K_N}\rho_{kj}^{NN}\left(\sigma_{ksT_{i-1}}^N - \sigma_{ksT_i}^N\right)\left(\sigma_{jsT_{N_i}}^N - \sigma_{jsT_i}^N\right)\\ &+\sum_{k=1}^{K_N}\sum_{j=1}^{K_R}\rho_{kj}^{NR}\left(\sigma_{jsT_i}^R - \sigma_{jsT_{i-1}}^R\right)\left(\sigma_{ksT_{N_i}}^N - \sigma_{ksT_i}^N\right)\end{aligned} \tag{9}$$

We now proceed to the valuation of LPI swaps.

Suppose that today, at time zero, we enter into an LPI swap. The LPI swap is defined via a set of fixed dates $T_0 < T_1 < T_2 < \cdots < T_{M-1} < T_M$, where $T_0 = 0$. The payment of the payout of the swap occurs at time T^*, where $T^* = T_M$. The payout of the inflation-linked leg of the swap at time T^* is given by:

$$\prod_{i=1}^{M}\min\left(\max\left(\frac{X_{T_i}}{X_{T_{i-1}}},1+F\right),1+C\right)$$

where C and F are constants with $C \geq F$. In practice, F is often zero but we will assume in the following that C and F can take on any values (positive, negative or zero) provided that $C \geq F$. We see that the role of the constants C and F is to cap and floor the period-on-period inflation rate over each period.

When $C = \infty$ and $F = -\infty$ the product 'telescopes' and the LPI swap has the same payout as a zero-coupon inflation swap. However, when C and F are finite and when $M > 1$, we need to price a swap whose payout is path-dependent. For typical values of M (between five and 40, say), the only feasible methodology to price LPI swaps is by Monte Carlo simulation, but this is computationally intensive. Hence, it would be desirable to have a fast, even if approximate, quasi-analytic methodology to price them. Such a methodology, based on the idea of common factor representation, is proposed in Ryten (2007). Note, however, that Ryten's model set-up is rather different from ours. We will apply Ryten's idea, in order to value LPI swaps, within the set-up of our multi-factor version of the Hughston (1998) and Jarrow & Yildirim (2003) model.

Let us begin by introducing some additional notation. We let $\mathbb{Q}_{T^*}$ denote the probability measure defined with respect to the numeraire that is the zero-coupon bond maturing at time T^*. Similarly, we let $\mathbb{E}_t^{T^*}[-]$ denote the expectation with respect to the measure $\mathbb{Q}_{T^*}$ conditional on $\{\mathcal{F}_t\}$. Suppose we have a T_M year LPI swap with M periods. Let X_i denote $X_{T_i}/X_{T_{i-1}}$ for $i = 1, 2, \ldots, M$. In Li (2007), it is shown that $\ln X_i$ for each $i = 1, 2, \ldots, M$ is normally distributed in our model, and that we can calculate the covariance matrix $\mathrm{cov}(\ln X_i, \ln X_j)$ for each i, j. In general, none of the elements of this covariance matrix vanish because $\ln X_i$ is not independent of $\ln X_j$ for any i, j. This lack of independence complicates the problem of pricing an LPI swap. The idea of Ryten (see also Jäckel, 2004) is to replace the covariance matrix $\mathrm{cov}(\ln X_i, \ln X_j)$ for each i, j by another matrix, which is close to the actual correlation matrix in some sense, but in which the off-diagonal elements have a simple structure. This is achieved by generating all the co-dependence between $\ln X_i$ and $\ln X_j$ through a single common factor (in fact, Ryten also considers the case of two common factors but we will, for the sake of brevity, only consider one).

It is easy to show (Li, 2007) that $\ln X_i = \ln X_{T_i}/\ln X_{T_{i-1}}$, $i = 1, 2, \ldots, M$, are distributed as multivariate normal random variables in the measure $\mathbb{Q}_{T^*}$. That is to say, $\ln X_i$ is Gaussian with deterministic drift and volatility under $\mathbb{Q}_{T^*}$. Hence, we can write X_i in the form $X_i = \exp(a_i z_i + b_i)$, where $z_i \sim N(0, 1)$; $\mathrm{cov}(\ln X_i, \ln X_j) = \mathrm{cov}(z_i, z_j)a_i a_j$; and $\mathbb{E}_t[X_i] = \exp(b_i + \frac{1}{2}a_i^2)$.

The key idea of Ryten (2007) is to replace X_i by $\hat{X}_i$ defined via:

$$\hat{X}_i \equiv \exp\left[b_i + a_i\left(\hat{a}_i w + \sqrt{\left(1-\hat{a}_i^2\right)}\varepsilon_i\right)\right]$$

where the system $\{w, \varepsilon_1, \ldots, \varepsilon_M\}$ is a family of independent $N(0, 1)$ variates. The variates $\hat{X}_1, \ldots, \hat{X}_M$ represent the variates $X_1, \ldots, X_M$ via one common factor w and additional individual idiosyncratic random variables $\{\varepsilon_i\}_{i=1,2,\ldots,M}$. Note that the common factor w is an abstract factor and does not necessarily correspond to any market-observable.

From Ryten (2007), which in turn references Jäckel (2004), we know that when $M \geq 3$ we can approximate $\hat{a}_k$ by:

$$\hat{a}_k \approx \exp\left[\frac{1}{M-2}\left(\bar{k}_k - \frac{\sum_{i=1}^{M}\bar{k}_i}{2(M-1)}\right)\right]$$

where $\bar{k}_k = \Sigma_{i\neq k}^{M}\ln[\mathrm{cov}(\ln X_i, \ln X_k)]$, $k = 1, 2, \ldots, M$. In the cases for which $M = 1$ or $M = 2$, we do not need an approximation. Indeed, if $M = 1$ then we have (trivially) $\hat{a}_1 = 1$; likewise if $M = 2$, then we have (from Cholesky decomposition) $\hat{a}_1 = 1$ and $\hat{a}_2 = \mathrm{Corr}(\ln X_1, \ln X_2)$.

Note that the relations $\mathbb{E}_0^{T^*}[\hat{X}_i] = \mathbb{E}_0^{T^*}[X_i]$ and $\mathrm{var}[\ln\hat{X}_i] = \mathrm{var}[\ln X_i]$ are valid for all $i = 1, 2, \ldots, M$ and for all values of M. However, if $M \geq 3$, then $\mathrm{cov}(\hat{X}_i, \hat{X}_j)$ is only an approximation to $\mathrm{cov}(X_i, X_j)$ when $i \neq j$.

We now apply Ryten's idea to value LPI swaps. By changing the measure to $\mathbb{Q}_{T^*}$ and using Girsanov's theorem, the price at time $T_0 = 0$ of the inflation-linked leg of the LPI swap is:

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left(-\int_0^{T^*} r_s^N\,ds\right)\prod_{i=1}^{M}\min\left(\max\left(\frac{X_{T_i}}{X_{T_{i-1}}}, 1+F\right), 1+C\right)\right]\\
&= P_{0T^*}^N\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M}\min\left(\max\left(\frac{X_{T_i}}{X_{T_{i-1}}}, 1+F\right), 1+C\right)\right]\\
&\approx P_{0T^*}^N\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M}\min\left(\max\left(\hat{X}_i, 1+F\right), 1+C\right)\right] \qquad (10)\\
&= P_{0T^*}^N\mathbb{E}_0^{T^*}\left[\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M}\min\left(\max\left(\hat{X}_i, 1+F\right), 1+C\right)\middle| w\right]\right]\\
&= P_{0T^*}^N\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M}\mathbb{E}_0^{T^*}\left[\min\left(\max\left(\hat{X}_i, 1+F\right), 1+C\right)\right]\middle| w\right]
\end{aligned}$$

By assumption the random variables ε_i are independent, and consequently, conditional on w, the variates $\hat{X}_i$ are also independent, that is, $\mathrm{cov}(\hat{X}_i, \hat{X}_j \mid w) = 0$, when $i \neq j$. Therefore, we see that the conditional expectation of the product in the last but one line of equation (10) becomes a product of conditional expectations in the last line. We have used $\approx$ (approximately equals) in the third line of equation (10) because the variates $\hat{X}_i$ are, in general (that is, when $M \geq 3$), only an approximate representation of the variates X_i for $i = 1, 2, \ldots, M$.

To evaluate equation (10) we need to calculate the $\mathbb{Q}_{T^*}$-expectation of X_i and the covariance matrix $\mathrm{cov}(\ln X_i, \ln X_j)$. The latter is shown in Li (2007) to be given by:

$$\begin{aligned}
&\mathrm{cov}\left(\ln X_i, \ln X_j\right)\\
&= \int_0^{T_{i-1}}\mathrm{cov}\left(\sum_{k=1}^{K_R}\left(\sigma_{ksT_i}^R - \sigma_{ksT_{i-1}}^R\right)dz_{ks}^R - \sum_{p=1}^{K_N}\left(\sigma_{psT_i}^N - \sigma_{psT_{i-1}}^N\right)dz_{ps}^N,\right.\\
&\qquad\left.\sum_{k=1}^{K_R}\left(\sigma_{ksT_j}^R - \sigma_{ksT_{j-1}}^R\right)dz_{ks}^R - \sum_{p=1}^{K_N}\left(\sigma_{psT_j}^N - \sigma_{psT_{j-1}}^N\right)dz_{ps}^N\right)ds\\
&+ \int_{T_{i-1}}^{T_i}\mathrm{cov}\left(\sigma_s^X dz_s^X + \sum_{k=1}^{K_R}\sigma_{ksT_i}^R dz_{ks}^R - \sum_{p=1}^{K_N}\sigma_{psT_i}^N dz_{ps}^N,\right.\\
&\qquad\left.\sum_{k=1}^{K_R}\left(\sigma_{ksT_j}^R - \sigma_{ksT_{j-1}}^R\right)dz_{ks}^R - \sum_{p=1}^{K_N}\left(\sigma_{psT_j}^N - \sigma_{psT_{j-1}}^N\right)dz_{ps}^N\right)ds
\end{aligned}$$

when $j > i$, whereas when $j = i$ we have:

$$\operatorname{var}(\ln X_i) \equiv \sigma^2_{\ln X_i}$$

$$= \int_0^{T_{i-1}} \operatorname{var}\left(\sum_{k=1}^{K_R} \left(\sigma^R_{ksT_i} - \sigma^R_{ksT_{i-1}} \right) dz^R_{ks} - \sum_{p=1}^{K_N} \left(\sigma^N_{psT_i} - \sigma^N_{psT_{i-1}} \right) dz^N_{ps} \right) ds$$

$$+ \int_{T_{i-1}}^{T_i} \operatorname{var}\left(\sigma^X_s dz^X_s + \sum_{k=1}^{K_R} \sigma^R_{ksT_i} dz^R_{ks} - \sum_{p=1}^{K_N} \sigma^N_{psT_i} dz^N_{ps} \right) ds$$

The former can also be calculated since it follows from the Girsanov theorem that the $\mathbb{Q}_{T^*}$-expectation of X_i is:

$$\mathbb{E}_0^{T^*}\left[\frac{X_{T_i}}{X_{T_{i-1}}} \right] = \frac{1}{P^N_{0T^*}} \mathbb{E}_0 \left[\exp\left(-\int_0^{T^*} r^N_s \, ds \right) \frac{X_{T_i}}{X_{T_{i-1}}} \right] \quad (11)$$

The $\mathbb{Q}_{T^*}$-expectation of X_i can then be evaluated explicitly by use of propositions 1 and 2. Specifically, when $i = 1$, we find, since $T_0 = 0$, that (11) implies:

$$\mathbb{E}_0^{T^*}[X_i] = \frac{P^R_{0T_1}}{P^N_{0T_1}} \exp\left(\int_0^{T_1} C_s\left(T_1, T^*\right) ds \right) \quad (12)$$

whereas when $i > 1$ we obtain:

$$\mathbb{E}_0^{T^*}[X_i] = \frac{P^N_{0T_{i-1}}}{P^N_{0T_i}} \frac{P^R_{0T_i}}{P^R_{0T_{i-1}}} \exp\left(\int_{T_{i-1}}^{T_i} C_s\left(T_i, T^*\right) ds + \int_0^{T_{i-1}} \left[A_s\left(T_{i-1}, T_i\right) + B_s\left(T_{i-1}, T_i, T^*\right) \right] ds \right) \quad (13)$$

Furthermore, since X_i is lognormal, we can use the standard result that if we denote by $\mu_{\ln X_i}$ and $\sigma^2_{\ln X_i}$ the mean and variance of $\ln X_i$, then $\mathbb{E}_0^{T^*}[X_i] = \exp(\mu_{\ln X_i} + ½\sigma^2_{\ln X_i})$ for $i = 1, 2, \ldots, M$. Hence we obtain the expectation of $\ln X_i$: $\mu_{\ln X_i} = \ln(\mathbb{E}_0^{T^*}[X_i]) - ½\sigma^2_{\ln X_i}$.

Now we can use the following well-known result: if $X \sim N(\mu_X, \sigma^2_X)$, $W \sim N(0, 1)$, and ρ_{XW} is the correlation between X and W, then $X \mid (W = w)$ is normally distributed and, furthermore, $\mathbb{E}[X \mid W = w] = \mu_X + \rho_{XW}\sigma_X w$ and $\operatorname{var}[X \mid W = w] = \sigma^2_X(1 - \rho^2_{XW})$.

We can calculate the correlation between $\ln\hat{X}_i$ and the common factor w. Indeed, since $\ln\hat{X}_i$ is normally distributed with variance a_i^2, and since:

$$\operatorname{cov}\left(\ln \hat{X}_i, w\right) = \operatorname{cov}\left(a_i\left(\hat{a}_i w + \sqrt{\left(1 - \hat{a}_i^2\right)}\varepsilon_i \right), w \right) = a_i \hat{a}_i$$

we deduce that the correlation between $\ln\hat{X}_i$ and w is $\hat{a}_i$ for each $i = 1, 2, \ldots, M$. Now we recall that $\mathbb{E}_0^{T^*}[\ln\hat{X}_i] = \mathbb{E}_0^{T^*}[\ln X_i] = \mu_{\ln X_i}$ and that $\operatorname{var}[\ln\hat{X}_i] = \operatorname{var}[\ln X_i] = \sigma^2_{\ln X_i}$. Then using the result above we get:

$$\mathbb{E}_0^{T^*}\left[\ln \hat{X}_i | w \right] = \mu_{\ln X_i} + \hat{a}_i \sigma_{\ln X_i} w$$

$$\bar{\sigma}_i^2 \equiv \operatorname{var}\left[\ln \hat{X}_i | w \right] = \sigma^2_{\ln X_i}\left(1 - \hat{a}_i^2\right)$$

and:

$$\bar{F}_i \equiv \mathbb{E}_0^{T^*}\left[\hat{X}_i | w \right] = \exp\left(\mu_{\ln X_i} + \hat{a}_i \sigma_{\ln X_i} w + \tfrac{1}{2}\bar{\sigma}_i^2 \right)$$

for $i = 1, 2, \ldots, M$.

Finally, equation (10) becomes:

$$P^N_{0T^*}\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M} \left(\bar{F}_i - Call\left(\bar{F}_i, 1 + C, \bar{\sigma}_i^2\right) + Put\left(\bar{F}_i, 1 + F, \bar{\sigma}_i^2\right) \right) \right] \quad (14)$$

where $Call(\bar{F}_i, 1 + C, \bar{\sigma}_i^2)$ and $Put(\bar{F}_i, 1 + F, \bar{\sigma}_i^2)$ are, respectively, the undiscounted prices of a call option with strike $1 + C$ and a put option with strike $1 + F$, in the Black (1976) formula, when the forward price is $\bar{F}_i$ and the integrated variance is $\bar{\sigma}_i^2$. Note that each term in the product in equation (14) depends on the common factor w through $\bar{F}_i$ and $\bar{\sigma}_i^2$, and w has a standard normal $N(0, 1)$ distribution. Hence, the price of the inflation-linked leg of the LPI swap at time zero (note that when $M \geq 3$, it is only an approximation) is:

$$P^N_{0T^*}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2} \right) \prod_{i=1}^{M} \left(\bar{F}_i - Call\left(\bar{F}_i, 1 + C, \bar{\sigma}_i^2\right) + Put\left(\bar{F}_i, 1 + F, \bar{\sigma}_i^2\right) \right) dw$$

It follows that we can value LPI swaps with just a single numerical integration.

Numerical examples

We now examine some numerical examples. There are different forms that the volatility functions σ^N_{ktT} and σ^R_{jtT} can take, but here we will consider the extended Vasicek form in which we assume:

$$\sigma^N_{ktT} = \frac{\sigma^N_k}{\alpha^N_k}\left(1 - e^{-\alpha^N_k (T-t)}\right), \quad \sigma^R_{ktT} = \frac{\sigma^R_k}{\alpha^R_k}\left(1 - e^{-\alpha^R_k (T-t)}\right) \quad (15)$$

where, for each k, σ^N_k, σ^R_k, α^N_k and α^R_k are positive constants.

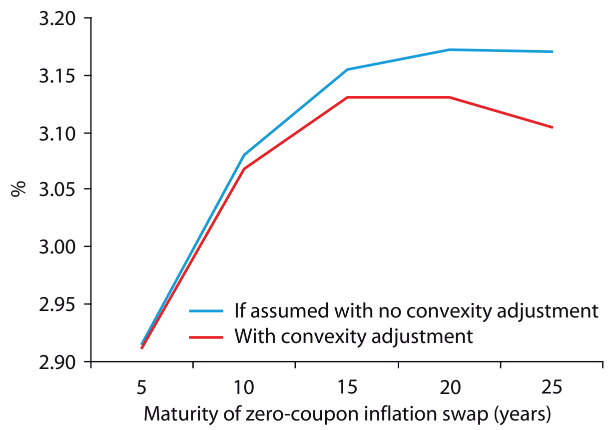
We will use the model parameters estimated for sterling in Li (2007). To simplify parameter estimation, we assume that real zero-coupon bond prices are driven by a single Brownian motion so that $K_R = 1$ in equation (3). In addition, we assume that the volatility of the spot CPI is constant, that is, $\sigma^X_t = \sigma^X$. We assume that there are two Brownian motions driving nominal zero-coupon bond prices so that $K_N = 2$. This assumption adds nothing to the complexity of the calibration, since the associated parameters can be (and were) obtained by calibrating to the market prices of sterling vanilla interest rate swaptions (Li, 2007). The estimated values of the parameters are:

$\sigma^N_1 = 0.00649825$	$\alpha^N_1 = 0.06494565$
$\sigma^N_2 = 0.0063321172$	$\alpha^N_2 = 0.00001557535$
$\sigma^R_1 = 0.006093904$	$\alpha^R_1 = 0.032193009$
$\sigma^X = 0.0104000$	$\rho^{NN}_{12} = -0.46296278$
$\rho^{RX}_1 = 0.03781752$	$\rho^{NR}_{11} = \rho^{NR}_{21} = 0.518100$
$\rho^{NX}_1 = \rho^{NX}_2 = 0.018398113$	

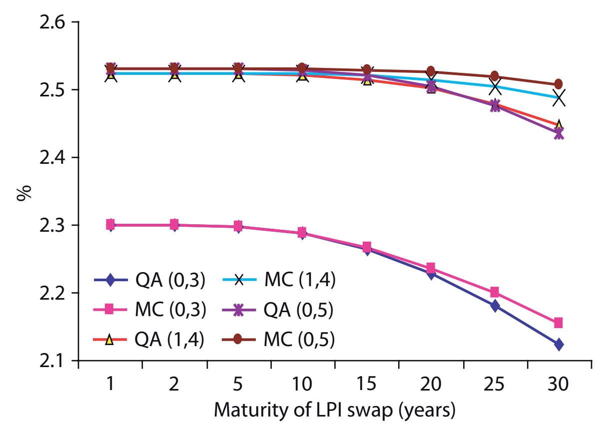
We will use these parameters to give some numerical examples and comparisons for inflation swaps with different swap tenors and payment times.

■ **Example 1: the effect of the convexity adjustment on the fixed rate for zero-coupon inflation swaps.** Figure 1 shows the fixed rate K on zero-coupon inflation swaps, with a payment delay of five years, for swaps of different tenors from five years to 25 years. The interest rate (both nominal and real) yield curves were the sterling market implied rates as of June 2007 (see Appendix B for the set of market data). The volatility and correlation parameters were as above. The fixed rate on the swaps when we evaluate the convexity adjustment, using proposition 1, is always lower than the fixed rate we would obtain on the swaps if we naively

1 Fixed rate K on zero-coupon inflation swaps with a payment delay of five years, for swaps of different maturities



2 Fixed rate on LPI swaps



A. Ten-year, 10-period LPI swap

Cap	Floor	Std. err.	Price Monte Carlo	Price Ryten (QA)	Imp. rate MC (%)	Imp. rate QA (%)	Diff. rates (%)
0.03	0	7.08E-06	0.760519	0.760461	2.28825	2.28746	0.00079
0.03	0.02	7.33E-06	0.777059	0.777044	2.50856	2.50836	0.00020
0.032	0.01	7.15E-06	0.767780	0.767724	2.38549	2.38475	0.00075
0.035	0.005	7.15E-06	0.770922	0.770840	2.42731	2.42622	0.00110
0.04	0.01	7.21E-06	0.778157	0.778063	2.52303	2.52179	0.00123
0.045	0.0175	7.33E-06	0.789247	0.789174	2.66821	2.66727	0.00094
0.0475	0.0025	7.21E-06	0.778593	0.778464	2.52878	2.52708	0.00170
0.05	0	7.21E-06	0.778669	0.778535	2.52978	2.52801	0.00177
0.05	0.005	7.21E-06	0.779410	0.779282	2.53953	2.53785	0.00169
0.06	0	7.21E-06	0.779061	0.778922	2.53493	2.53311	0.00183
0.12	−0.08	7.21E-06	0.778796	0.778654	2.53145	2.52957	0.00188

B. Twenty-five-year, 25-period LPI swap

Cap	Floor	Std. err.	Price Monte Carlo	Price Ryten (QA)	Imp. rate MC (%)	Imp. rate QA (%)	Diff. rates (%)
0.03	0	1.62E-05	0.493509	0.491246	2.19897	2.18018	0.01879
0.03	0.02	1.87E-05	0.530458	0.529970	2.49455	2.49077	0.00378
0.032	0.01	1.69E-05	0.509992	0.508136	2.33336	2.31844	0.01492
0.035	0.005	1.66E-05	0.514297	0.511382	2.36778	2.34451	0.02327
0.04	0.01	1.71E-05	0.531668	0.528436	2.50389	2.47889	0.02500
0.045	0.0175	1.82E-05	0.557735	0.555077	2.70033	2.68071	0.01962
0.0475	0.0025	1.66E-05	0.533227	0.528129	2.51590	2.47651	0.03939
0.05	0	1.65E-05	0.533657	0.528121	2.51920	2.47645	0.04275
0.05	0.005	1.67E-05	0.536505	0.531414	2.54103	2.50193	0.03910
0.06	0	1.65E-05	0.536619	0.530530	2.54190	2.49511	0.04680
0.12	−0.08	1.63E-05	0.535254	0.528384	2.53145	2.47849	0.05297

assumed that no convexity adjustment was necessary. Furthermore, the difference increases with increasing swap tenor. At 25 years, that is, when $T_M = 25$ and $T_N = 30$, the difference is more than 0.065%, which is significant from a trader's perspective, as the bid-offer spread in the market, for zero-coupon inflation swaps, is approximately 0.03%, or sometimes even less.

Some examples of period-on-period inflation swaps are provided in Li (2007) so here, in examples 2 and 3, we will give some examples of the prices of LPI swaps, again using the volatility and correlation parameters above. For the purposes of these illustrations, we assumed, for both the examples below, that the interest rate (both nominal and real) yield curves were initially flat and that nominal interest rates to all maturities were 0.05 and real interest rates to all maturities were 0.025, that is, we assumed $P^N_{0T} = \exp(-0.05T)$ and $P^R_{0T} = \exp(-0.025T)$. We used Monte Carlo simulation with 130 million runs (65 million runs plus 65 million antithetic runs) to test and benchmark the accuracy of our application of the Ryten methodology.

■ **Example 2: LPI swaps with floors and caps at (0%, 3%), (0%, 5%) and (1%, 4%).** Here we consider three different combinations of floors and caps (which are commonly traded in the market), namely (0%, 3%), (0%, 5%) and (1%, 4%). For all three different combinations, we consider LPI swaps where each period is equal to one year, and the number of periods varies from one period, through two, five, 10, 15, 20, 25 and 30 periods, and hence the maturities of the LPI swaps varied from one year to 30 years. We see from figure 2 that the fixed rates obtained from the quasi-analytical methodology of Ryten (QA) are very close to those obtained from Monte Carlo (MC) simulation for shorter maturities (as explained above, the Ryten methodology is, in fact, essentially exact for $M \leq 2$). However, the differences do increase for LPI swaps with more periods.

■ **Example 3: LPI swaps with maturities of 10 years and 25 years.** Here we consider 11 different combinations of floors and caps as shown in table A. We consider LPI swaps whose maturities were 10 years and 25 years. Again, each period is equal to one year. We know that the Ryten methodology is essentially exact when $M \leq 2$. However, we see for the LPI swaps with 10 years' maturity and 25 years' maturity the level of approximation involved when $M \geq 3$. As a rough guide, the bid-offer spread in the market for LPI swaps is approximately 0.06% (expressed as the fixed rate on the swap). For the LPI swaps with 10 years' maturity, the maximum difference (table A, eighth column) between the fixed rates implied by the Monte Carlo results (sixth column) and the Ryten methodology (seventh column) is less than 0.0019%, which implies very accurate pricing as it is less than one-thirtieth of the typical bid-offer spread. For the LPI swaps with 25 years' maturity, the accuracy does deteriorate somewhat. The maximum difference in the fixed rates is approximately 0.053%, which is close to the bid-offer spread.

Having given some examples of the valuation of LPI swaps, we can make one further comment about the accuracy of the quasi-analytical methodology. In tables A and B, we observe that the accuracy deteriorates when the cap level is high and the floor level is low. This might initially seem surprising since in the limiting case that $C = \infty$ and $F = -\infty$ the LPI swaps become the same as standard zero-coupon swaps. However, the reason for the deterioration in accuracy is that the quasi-analytical methodology approximates the correlation structure. Although (in the notation of the previous section) it is true that $\mathbb{E}_0^{T*}[\hat{X}_i] = \mathbb{E}_0^{T*}[X_i]$ for all i, and it is also true that $\mathbb{E}_0^{T*}[\Pi_{i=1}^{M} X_i] = \mathbb{E}_0^{T*}[X_{T_M}/X_0] = P^R_{0T_M} = P^R_{0T*}$, the price of a standard zero-coupon swap, the approximation of the correlation structure means that $\mathbb{E}_0^{T*}[\Pi_{i=1}^{M} \hat{X}_i]$ does not equal $\mathbb{E}_0^{T*}[\Pi_{i=1}^{M} X_i]$, except in the special cases for which $M \leq 2$.

For the sake of brevity, we only considered the Ryten methodology for the case of conditioning on one common factor. Ryten (2007) also considers the case of conditioning on two common factors (which means that evaluating the price of a LPI swap requires a double numerical integration) and shows, in his model set-up, which is different from ours, that (unsurprisingly) this gives a significant improvement in accuracy. We would certainly conjecture that using two common factors would also significantly improve the accuracy of the prices of the LPI swaps that we reported in tables A and B. However, we leave confirmation of this conjecture for future research.

Conclusion

In recent years, there has been a substantial increase in demand for more exotic inflation derivatives. Working within a multi-factor version of the model of Hughston (1998) and Jarrow & Yildirim (2003), we have provided the economic rationale for, and the valuation formulas for, zero-coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments. We have also valued LPI swaps, with the aid of the quasi-analytic methodology of Ryten (2007). ■

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Appendix A: proof of propositions 1 and 2

The stochastic discounting term $\exp\left(-\int_t^{T_N} r_s^N ds\right)$ is lognormally distributed and can be written in the form:

$$\exp\left(-\int_t^{T_N} r_s^N ds\right)$$
$$= P_{tT_N}^N \exp\left(-\int_t^{T_N} \frac{1}{2}\sum_{k=1}^{K_N}\sum_{j=1}^{K_N} \rho_{kj}^{NN} \sigma_{ksT_N}^N \sigma_{jsT_N}^N ds\right)$$
$$\times \exp\left(\int_t^{T_N} \sum_{k=1}^{K_N} \sigma_{ksT_N}^N dz_{ks}^N\right)$$

If we define the forward consumer price index at time t to time T by F_{tT}^X, then by no-arbitrage arguments, we have $F_{tT}^X = X_t(P_{tT}^R/P_{tT}^N)$, where F_{tT}^X is lognormally distributed (see, for example, Crosby, 2007). Since:

$$F_{T_M T_M}^X = X_{T_M} \frac{P_{T_M T_M}^R}{P_{T_M T_M}^N} = X_{T_M}$$

we find:

$$\mathbb{E}_t\left[\exp\left(-\int_t^{T_N} r_s^N ds\right) X_{T_M}\right]$$
$$= \mathbb{E}_t\left[\exp\left(-\int_t^{T_N} r_s^N ds\right) F_{T_M T_M}^X\right]$$

This expectation can be calculated by noting that it is the expectation of a product of two lognormally distributed random variables, each of which has deterministic mean and variance terms. Li (2007) provides full details.

The proof of proposition 2 is very similar to that for proposition 1 except that we will calculate an expectation involving three lognormally distributed random variables.

Appendix B: market data for example 1

Tenor	Nominal discount factors	Real discount factors
5	0.747665196	0.863178385
10	0.574072261	0.777518375
15	0.450566319	0.717981039
20	0.361027914	0.674313663
25	0.301528182	0.657905735
30	0.242028449	0.614217677

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